Theory of Fuzzy Sets

An introduction to fuzzy set theory, as well as, studying some of its basic and fundamental concepts with some properties, which are necessary for understanding the subject of fuzzy sets will be given in this part. Also, several examples are considered for illustration purpose.

1. History

In basic sciences, such as engineering, chemistry or physics, we construct exact or precise mathematical models of empirical phenomena, and then using these models to make predictions. While some aspects of the real world problems always escape from such precise models, and usually there is an elusive inexactness as a part of the original model. Also, elements of the real world problems are perturbed by imperfection and thus, for example, there exist no elements that are perfectly round. Perfect notions or exact concepts correspond to the sort of things envisaged in pure mathematics, while inexact structures encounter us in real life problems.

The fuzzy set theory was initiated by L. A. Zadeh in the early of 1960's especially in 1965. However, the term ensemble flou "a posterior the French counterpart of fuzzy set" was coined by Menger in 1951. Menger explicitly used the maximum product in transitive fuzzy relation, but with a probabilistic interpretation. On a semantic level Zadeh's theory is more closely to Blank's work in vagueness.

Historically, the general accepted birth date of the theory of fuzzy sets is 1965, when the first article entitled "Fuzzy Sets" given by Zadeh appeared in the journal of information and control. Also, the term "fuzzy" was introduced and coined by Zadeh in this article.

2. Basic Concepts of Fuzzy Sets

In this section, some basic definitions and fundamental concepts related to fuzzy set theory are given including some basic algebraic operations defined in fuzzy sets. Additional concepts may be found in any text book concerning fuzzy sets.

We start this section by the definition of ordinary or nonfuzzy sets in order to give a comparison with fuzzy sets, and to give the reason for the introduction of fuzzy sets.

Definition (2.1):

Let X be a classical set of objects, called the universal set, whose generic elements are denoted by x. The membership in a classical subset A of X is often viewed as a characteristic function χ from X into {0, 1}, such that:

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

The set {0, 1} is called a valuation set.

Fuzzy sets will be distinguished from nonfuzzy sets using the symbol "~", i.e., fuzzy sets will be abbreviated as $\tilde{A}$ and nonfuzzy sets are denoted by A.

Definition (2.2):

Let X be the universal set and $\tilde{A}$ be any subset of X, then $\tilde{A}$ is called fuzzy subset of X, which is characterized by a membership function $\mu_{\tilde{A}} : X \longrightarrow [a, b]$, where $a, b \in \mathbb{R}$ and in a special case $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$, i.e.,

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}.$$

Remark (2.3):

For simplicity, the collection of all fuzzy subsets of X will be denoted by I^X or X*, i.e.,

$$I^X = \{\tilde{A} : \tilde{A} \text{ is a fuzzy subsets of X}\}.$$

where I = [0, 1].

The basic concepts with logical and algebraic operations defined on fuzzy sets may be summarized in the next remark:

Remarks (2.4):

Let X be the universal set, and $\tilde{A}$, $\tilde{B}$ be two fuzzy subsets of X with membership function $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$, respectively; then:

1. X may be considered as a fuzzy set with membership function $\mu_{\tilde{A}}(x) = 1$, for all $x \in X$, which is denoted by 1_X; while the empty fuzzy set $\tilde{\varnothing}$ is a fuzzy set with membership function $\mu_{\tilde{\varnothing}}(x) = 0, \forall\, x \in X$, which is denoted by 0_X.

2. The elements of X, such that $\mu_{\tilde{A}}(x) = \frac{1}{2}$ are called the crossover points of $\tilde{A}$.

3. The height of $\tilde{A}$ is the greatest membership value, i.e.,

$$\text{hgt}(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x)$$

4. $\tilde{A}$ is said to be normal if there exists $x_0 \in X$, such that $\mu_{\tilde{A}}(x_0) = 1$, otherwise $\tilde{A}$ is subnormal. Also, if a fuzzy set $\tilde{A}$ is subnormal, then it may be normalized by dividing $\mu_{\tilde{A}}$ on $\text{hgt}(\tilde{A}) \neq 0$.

5. The ordinary or nonfuzzy set $\{x : x \in X \mid \mu_{\tilde{A}}(x) > 0\}$ is called the support of $\tilde{A}$ and is denoted by $\text{Supp}(\tilde{A})$.

6. $\tilde{A}^c$ is the complement of $\tilde{A}$ which is also a fuzzy set with membership function:

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x), \forall\, x \in X.$$

7. $\tilde{A} \subseteq \tilde{B}$ if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall\, x \in X$.

8. $\tilde{A} = \tilde{B}$ if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \forall\, x \in X$.

9. The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{C}$ and may be associated with the following membership function:

$$\mu_{\tilde{C}}(x) = \text{Max}\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}, \quad \forall\, x \in X$$

10. The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{D}$ and may be defined with the following membership function:

$$\mu_{\tilde{D}}(x) = \text{Min}\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}, \ \forall\ x \in X$$

More generally, for a family of fuzzy subsets of I^X, then the union $\tilde{C} = \bigcup_i \tilde{A}_i$, and the intersection, $\tilde{D} = \bigcap_i \tilde{A}_i$, are fuzzy sets with the membership functions defined by:

$$\mu_{\tilde{C}}(x) = \sup_i \{\mu_{\tilde{A}_i}(x) : x \in X\}$$

$$\mu_{\tilde{D}}(x) = \inf_i \{\mu_{\tilde{A}_i}(x) : x \in X\}$$

11. The addition of $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{C}$ with some membership function:

$$\mu_{\tilde{C}}(x) = \min\{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x), 1\}, \ \forall\ x \in X$$

12. The subtraction of $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{C}$ with a suitable membership function:

$$\mu_{\tilde{C}}(x) = \max\{0, \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)\}, \ \forall\ x \in X$$

13. The algebraic product of $\tilde{A}$ and $\tilde{B}$ (denoted by $\tilde{A} \odot \tilde{B}$) is a fuzzy set $\tilde{C}$ with membership function:

$$\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x)\, \mu_{\tilde{B}}(x), \ \forall\ x \in X.$$

14. The probabilistic sum of $\tilde{A}$ and $\tilde{B}$ (denoted by $\tilde{A} \oplus \tilde{B}$) is a fuzzy set $\tilde{C}$ with membership function:

$$\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x), \ \forall\ x \in X.$$

15. $\tilde{A}$ and $\tilde{B}$ are said to be separated if $\tilde{A} \cap \tilde{B} = \tilde{\varnothing}$.

16. A fuzzy subset $\tilde{A}$ of $\mathbb{R}$ is said to be convex fuzzy set, if:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \text{Min}\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right\}$$

for all x_1, $x_2 \in \mathbb{R}$, and all $\lambda \in [0, 1]$, where $\mu_{\tilde{A}}(x)$ is standing for a suitable membership function

17. Following [Gerla G., 1998], let X be the Cartesian product of universes X_1, X_2, …, X_r and $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_r$ be r-fuzzy subsets of X_1, X_2, …, X_r, respectively. Let f be a mapping from X to a universe Y. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y, by:

$$\tilde{B} = f(\tilde{A})$$

$$= \left\{ \left(y, \mu_{\tilde{B}}(y) \right) \middle| \; y = f\left(x_1, x_2, \ldots, x_r\right), \left(x_1, x_2, \ldots, x_r\right) \in X \right\}$$

where:

$$\mu_{\tilde{B}}\left(y\right) = \begin{cases} \sup\limits_{(x_1, x_2, \ldots, x_r) \in f^{-1}(y)} \min\left\{ \mu_{\tilde{A}_1}(x_1), ..., \mu_{\tilde{A}_r}(x_r) \right\}, & \text{if } f^{-1}\left(y\right) \neq \varnothing \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , & \text{Otherwise} \end{cases}$$

where f^{-1} is the inverse image of f.

For r = 1, the extension principle, reduced to:

$$\tilde{B} = f(\tilde{A}) = \left\{ (y, \mu_{\tilde{B}}(y)) \middle| \; y = f(x), \, x \in X \right\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if } f^{-1}\left(y\right) \neq \varnothing \\ 0 \qquad\qquad , & \text{Otherwise} \end{cases}$$

which is the definition of a fuzzy function.

18. Let X and Y be two nonempty sets and f : X $\longrightarrow$ Y be a mapping. The inverse image of a fuzzy set $\tilde{C}$ in Y is defined also using the extension principle, as follows:

$$(f^{-1}(\tilde{C}))(x) = \mu_{\tilde{C}}(f(x))$$

for all $x \in X$.

19. The inverse image is reduces to inverse function, where the function becomes 1-1 and onto, and that will change the definition of extension principle.

Remark (2.5):

A more convenient notation is proposed by Zadeh is as follows, if μ_i is the grad of the membership function of $x_i \in X$ then $\mu_i|x_i$ is termed as a fuzzy singleton. Hence in terms of the constituent singleton of the fuzzy set $\tilde{A}$ with its finite number of elements $x_1, x_2, ..., x_n$ can also be expressed in union form by:

$$\tilde{A} = \mu_1|x_1 + \mu_2|x_2 + \ldots + \mu_n|x_n$$

$$= \sum_{i=1}^{n} \mu_i|x_i$$

$$= \bigcup_{i=1}^{n} \mu_i|x_i$$

and when X is uncountable , we use:

$$\tilde{A} = \int_{x \in X} \mu_x|x$$

where the integral sign stands for the union (not the integration operator) of the fuzzy singletons .

Following are some illustrative examples which explain the necessity of the fuzzy sets.

Example (2.7):

Let $X = \mathbb{R}^+$ be the set of possible ages for a human beings, where $\mathbb{R}^+$ is the set of all positive real numbers, then the fuzzy set:

$\tilde{C}$ = "About 50 years old"

may be expressed as:

$$\tilde{C} = \{(x, \mu_{\tilde{C}}(x)) \mid x \in X\}$$

with membership function:

$$\mu_{\tilde{C}}(x) = \frac{1}{1 + \left(\frac{x - 50}{10}\right)^4}$$

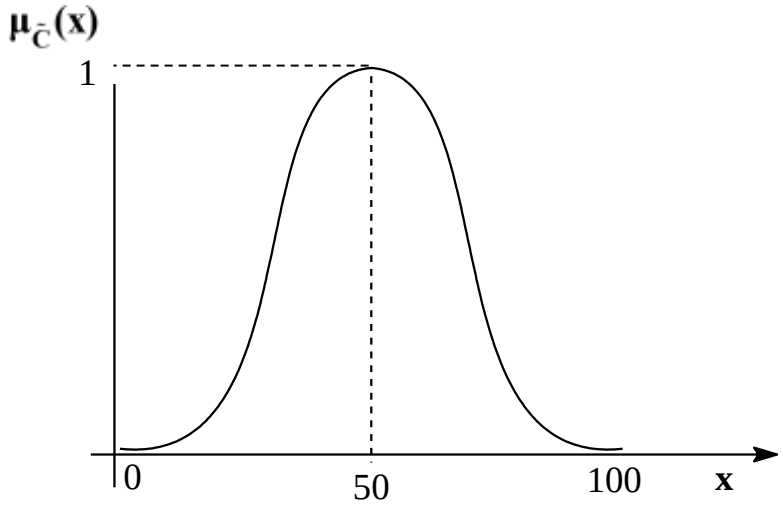


Fig.(1) Membership function on a continuous universe of example (2.7).

3. α-Cuts or α-Sets

In this section, an important notion in fuzzy sets will be discussed and introduced, which is the so called the α-cut sets, which corresponds to any fuzzy set $\tilde{A}$. α-Cuts sets, may be considered as an intermediate set that connect between fuzzy sets and ordinary sets, which may be used to prove that most of the results that are satisfied in ordinary sets are also satisfied here to fuzzy sets.

In fuzzy set theory, if one wants to exhibit an element $x \in X$ that is typically belong to a fuzzy set $\tilde{A}$, its membership value may be demanded to be greater than some threshold level $\alpha \in (0, 1]$. The ordinary set of such elements is called the α-cut sets of $\tilde{A}$ and is denoted by A_α, i.e.,

$$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha, \alpha \in (0, 1]\}$$

and the strong α-cut set:

$$A_{\alpha+} = \{x \in X : \mu_{\tilde{A}}(x) > \alpha, \alpha \in (0, 1]\}$$

An important notion used in the representation of a fuzzy set $\tilde{A}$ in terms of its elements or points, which is also fuzzy, is given in the next definition:

Remarks (3.1):

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of a universal set X, then the following properties are satisfied for all $\alpha \in (0, 1]$:

1. $\tilde{A} = \tilde{B}$ if and only if $A_\alpha = B_\alpha$, $\forall\ \alpha \in (0, 1]$
2. $\tilde{A} \subseteq \tilde{B}$ if and only if $A_\alpha \subseteq B_\alpha$, $\forall\ \alpha \in (0, 1]$.
3. If $\alpha \leq \beta$, then $A_\alpha \supseteq A_\beta$, $\forall\ \alpha, \beta \in (0, 1]$.
4. $A_\alpha \cap A_\beta = A_\beta$ and $A_\alpha \cup A_\beta = A_\alpha$, if $\alpha \leq \beta$, $\forall\ \alpha, \beta \in (0, 1]$.
5. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
6. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.

Examples (3.2):

Consider the universal set:

$X = \{a, b, c, e\}$

A fuzzy subset $\tilde{A}$ of X may be defined using the following membership function:

$$\mu_{\tilde{A}}(e) = 1/2,\ \mu_{\tilde{A}}(a) = 1/3,\ \mu_{\tilde{A}}(b) = 1/4,\ \mu_{\tilde{A}}(c) = 1/4$$

Hence the image set of $\tilde{A}$ (denoted by $\text{Im}(\tilde{A})$) is given by:

$$\text{Im}(\tilde{A}) = \{1/2, 1/3, 1/4\}$$

with the level sets:

$$A_{1/2} = \{e\},\ A_{1/3} = \{e, a\},\ A_{1/4} = \{e, a, b, c\}$$

Also, define another fuzzy set $\tilde{B}$ of X with membership function:

$$\mu_{\tilde{B}}(e) = 4/5,\ \mu_{\tilde{B}}(a) = 2/5,\ \mu_{\tilde{B}}(b) = 1/5,\ \mu_{\tilde{B}}(c) = 1/5$$

which has the image set:

$$\text{Im}(\tilde{B}) = \{4/5, 2/5, 1/5\}$$

with the level sets:

$$B_{4/5} = \{e\},\ B_{2/5} = \{e, a\},\ B_{1/5} = \{e, a, b, c\}$$

Thus, the fuzzy sets $\tilde{A}$ and $\tilde{B}$ have the same family of level sets, i.e.,

$A_{1/2} = B_{4/5}$, $A_{1/3} = B_{2/5}$, $A_{1/4} = B_{1/5}$

But, $\tilde{A}$ and $\tilde{B}$ are different, where:

$\tilde{A} = \{(e, 1/2), (a, 1/3), (b, 1/4), (c,1/4)\}$

$\tilde{B} = \{(e, 4/5), (a, 2/5), (b, 1/5), (c, 1/5)\}$.

Therefore, the necessary condition to ensure that $\tilde{A} = \tilde{B}$ is that they must have the same image set.

4. Fuzzy Number

Another type of fuzzy sets that may be characterized using different notion is the fuzzy subsets of real numbers, which is the so called fuzzy number. Simple examples of fuzzy numbers are fuzzy subset of the real line labeled small, approximately 8, very close to 5, more or less large, much larger than 6, etc.

Fuzzy numbers are always fuzzy sets while the converse is not true. The definition of a fuzzy number is as follows:

Definition (4.1):

A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set of the real line $\mathbb{R}$ and there exists exactly one $x_0 \in \mathbb{R}$, with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$) and $\mu_{\tilde{M}}(x)$ is piecewise continuous.

It is remarkable that a fuzzy number equals 1 at only one point $x_0 \in \mathbb{R}$ while a fuzzy interval equals 1 at an interval $[a, b] \subset \mathbb{R}$, and hence every fuzzy number is a fuzzy interval with $a = b = x_0$, but the converse is not true.

Since the fuzzy number is a fuzzy set, therefore the definition of the α-level sets may be applied here for fuzzy numbers as it is seen in the next definition.

Definition (4.2):

A fuzzy number $\tilde{M}$ may be uniquely represented in terms of its α-level sets, as the following closed intervals of the real line:

$M_\alpha = [m - \sqrt{1-\alpha}\,,\ m + \sqrt{1-\alpha}\,],\ \alpha \in (0, 1]$

where m is the mean value of $\tilde{M}$. This fuzzy number may be written as $M_\alpha = [\underline{\tilde{M}}, \overline{\tilde{M}}]$, where $\underline{\tilde{M}}$ refers to the greatest lower bound of M_α and $\overline{\tilde{M}}$ to the least upper bound of M_α, for certain $\alpha \in (0, 1]$.

Example (4.3):

The fuzzy number $\tilde{2}$ is the set of all $x \in \mathbb{R}$, with membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} x-1, & \text{if } 1 \le x < 2 \\ 3-x, & \text{if } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

which may be viewed as a fuzzy set and hence the α-level sets related to this fuzzy number are given by:

$2_\alpha = [2 - \sqrt{1-\alpha}\,,\ 2 + \sqrt{1-\alpha}\,]$, for all $\alpha \in (0, 1]$

For example:

When $\alpha = 0$, implies $2_0 = [2 - \sqrt{1-0}\,,\ 2 + \sqrt{1-0}\,] = [1, 3]$

When $\alpha = 0.5$, implies $2_{0.5} = [2 - \sqrt{1-0.5}\,,\ 2 + \sqrt{1-0.5}\,] = [1.293, 2.707]$

When $\alpha = 1$, implies $2_1 = [2 - \sqrt{1-1}\,,\ 2 + \sqrt{1-1}\,] = [2, 2] = 2$ (mean value of $\tilde{2}$), which is the crisp fuzzy number.

It is clear that $2_0 \supset 2_{0.5} \supset 2_1$ as one can see from the properties of the α-level sets.

Example (4.4):

The following fuzzy sets are fuzzy numbers:

Approximately $\tilde{5} = \{(3, 0.2), (4, 0.6), (5, 1), (6, 0.7), (7, 0.1)\}$

Approximately $\tilde{10} = \{(8, 0.3), (9, 0.7), (10, 1), (11, 0.7), (12, 0.3)\}$

But {(3, 0.8), (4, 1), (5, 1), (6, 0.7)} is not a fuzzy number; because 4 and 5 has membership values equals 1.

Remarks (4.5):

If the algebraic operations +, –, . and ÷ are extended to an operation between fuzzy numbers, then they will be denoted by $\tilde{+}$, $\tilde{-}$, $\tilde{\cdot}$ and $\tilde{\div}$, respectively, which may be defined using the extension principle (given in remark (2.4)(17)), as follows:

Let $\tilde{M}$ and $\tilde{N}$ be two fuzzy sets defined on the universal set X with membership functions $\mu_{\tilde{M}}$ and $\mu_{\tilde{N}}$, respectively and let * be one of the algebraic operations +, –, ., ÷, then $\tilde{+}$, $\tilde{-}$, $\tilde{\cdot}$ and $\tilde{\div}$ (which is denoted by o) between the two fuzzy numbers $\tilde{M}$ and $\tilde{N}$ will give a fuzzy set with membership function:

$$\mu_{\tilde{M}o\tilde{N}}(z) = \sup_{z=x*y} \min\{\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)\}, \forall\, x, y \in X$$

Example (4.6):

Consider the universal set X = {1, 2, …, 6} and define two fuzzy subsets of X, by:

$$\tilde{A} = \{(1, 0.3), (2, 0.7), (3, 0.2)\}$$

$$\tilde{B} = \{(1, 0.9), (2, 0.5), (3, 0.1)\}$$

Then $\tilde{C} = \tilde{A} \tilde{+} \tilde{B}$ is a fuzzy set with membership function:

$$\mu_{\tilde{C}}(z) = \sup_{z=x+y} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$$

Therefore, for z = 4, we have:

$$\mu_{\tilde{C}}(4) = \sup\{\min\{\mu_{\tilde{A}}(1), \mu_{\tilde{B}}(3)\}, \min\{\mu_{\tilde{A}}(2), \mu_{\tilde{B}}(2)\}, \min\{\mu_{\tilde{A}}(3), \mu_{\tilde{B}}(1)\}\}$$

$$= \sup\{0.1, 0.5, 0.2\} = 0.5$$

and so is similarly for z = 2, 3, 5, 6; which give finally the fuzzy set:

$$\tilde{C} = \{(2, 0.3), (3, 0.7), (4, 0.5), (5, 0.2), (6, 0.1)\}$$

5. The Membership Function

An important notion of the theory of fuzzy sets is the definition and construction of the membership functions, which admits certain properties of fuzzy sets. Therefore, in this section, the construction of such functions will be discussed in details. The characteristic function assigns to each element x of X a number, $\mu_{\tilde{A}}(x)$, in the closed unit interval [0, 1] that characterizes the degree of membership of x in $\tilde{A}$. In defining the membership function, the universal set X always assumed to be classical set (either discrete or continuous).

Two approaches may be used to define the membership function of a given fuzzy set; namely numerical approach or tabulated approach and theoretical or functional approach.

A numerical definition expresses the degree of membership function of a fuzzy set as a vector of numbers whose dimension depends on the level of discretization, i.e., the number of discrete elements in the universal set X. This method has some advantages and disadvantages, which are in advantage case its simplicity of construction and the disadvantage encountered in its very consuming and long definition, especially with those sets of so many elements (see example (2.8)).

Functional definition defines the membership function of a fuzzy set in an analytic expression, which allows the membership grade for each element in the defined universe of discourse to be calculated. Certain standard families or 'shapes' of membership functions are commonly used for fuzzy sets based on the universe of real numbers and on the definition of the fuzzy set.

Among the most common membership functions, which are often used in practice, include the following types:

1. Any symmetric, triangular shaped membership function used to define fuzzy numbers, which is characterized by the three parameters a, b and s, where a, $s \in \mathbb{R}$ ($s \neq 0$) and $0 < b \leq 1$, as shown in Fig.(2), is represented by the generic form:

$$\mu_{\tilde{A}}(x) = \begin{cases} b\left(1 - \dfrac{|x-a|}{s}\right), & \text{when } a - s \le x \le a + s \\ 0, & \text{otherwise} \end{cases}$$

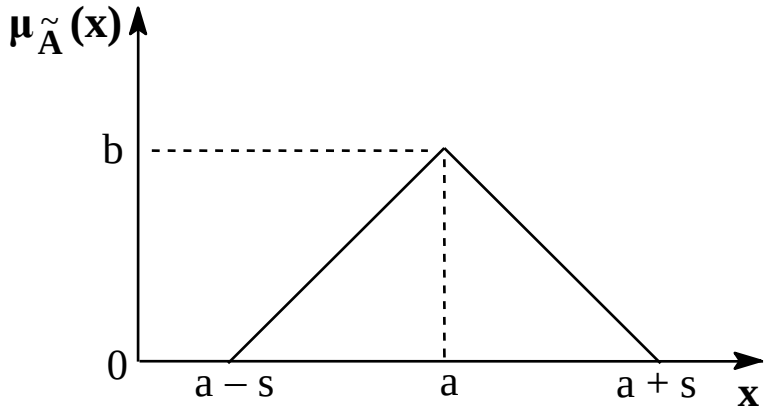


Fig.(2) The triangular membership functions.

2. Another important class of membership functions used to define fuzzy intervals is of trapezoidal shaped, which is captured by the generic graphical representation in Fig.(3). Each function in this class is fully characterized by the five parameters a, b, c, d and e, where a, b, c, d $\in \mathbb{R}$ and $0 < e \le 1$, via the general form:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(a-x)e}{a-b}, & \text{when } a \le x < b, a \ne b \\ e, & \text{when } b \le x < c \\ \dfrac{(d-x)e}{d-c}, & \text{when } c \le x < d, d \ne c \\ 0, & \text{otherwise} \end{cases}$$

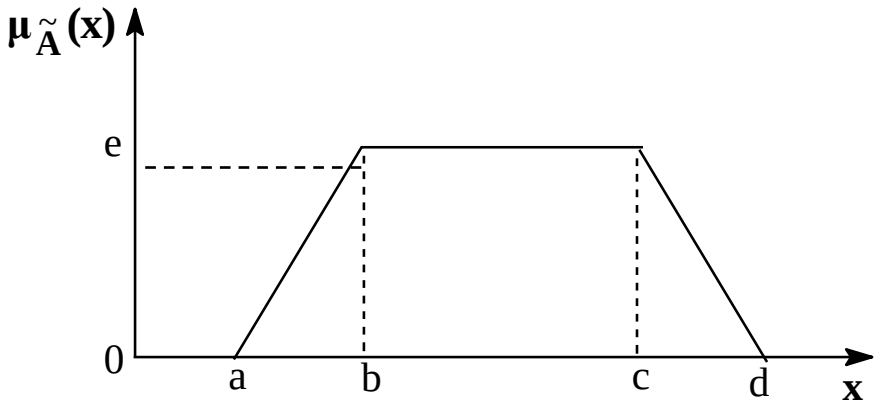


Fig.(3) The trapezoidal membership function.

3. Bell-shaped membership functions are also quite common in defining fuzzy numbers. A generic graph is shown in Fig.(1.4). These functions are presented by the formula:

$$\mu_{\tilde{A}}(x) = c\, e^{\frac{(x-a)^2}{b}}, \; b \neq 0$$

which involves three parameters a, b, and c, where $a \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$ and $0 \leq c \leq 1$, whose rules are indicated in Fig.(1.4).

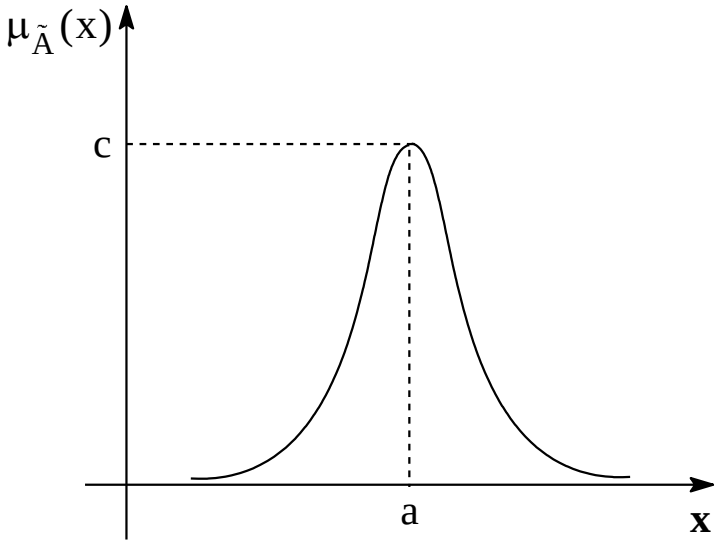


Fig.(1.4) Bell-shaped fuzzy set.

4. The S-membership function is defined as follows:

$$S(x; a, b, c) = \begin{cases} 0 & \text{for } x < a \\ 2\left[(x-a)/(c-a)\right]^2 & \text{for } a \leq x < b \\ 1 - 2\left[(x-c)/(c-a)\right]^2 & \text{for } b \leq x < c \\ 1 & \text{for } x > c \end{cases}$$

Functions in this family have an "S" shape whose precise appearance is determined by the value of the parameters of a, b, c, as illustrated in Fig.(5). Note that the S-function is flat with constant value 0 for x < a; and constant value 1 for x > c. In between of a and c, the S-function is a quadratic function of x, b = (a + c)/2.

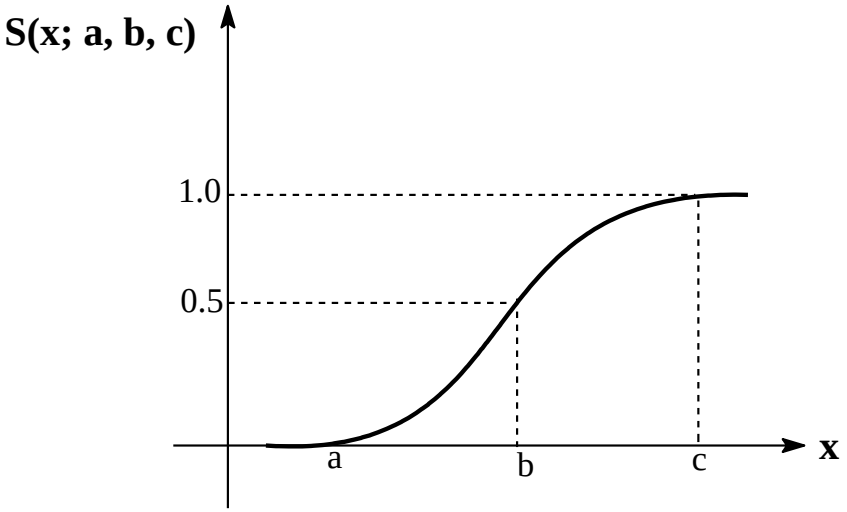


Fig.(1.5) The S-Function.

5. The π-function which may be used also to define fuzzy numbers is defined by:

$$\pi(x;\, b,\, c) = \begin{cases} S(x; c-b, c-b/2, c) & \text{for } x \le c \\ 1-S(x; c, c+b/2, c+b) & \text{for } x > c \end{cases}$$

Functions in this family are also of bell- shaped, with the sides of the bell being generated from the S-functions. Functions of this type may be used as an alternative to the triangular-functions as they give a membership value, which approaches 0 in a more gradual manner, as illustrated in Fig.(6). Note that the b parameter is now the bandwidth at the crossover points. The π-function goes to zero at the points $x = c \pm b$, while the crossover points are at $x = c \pm b/2$.

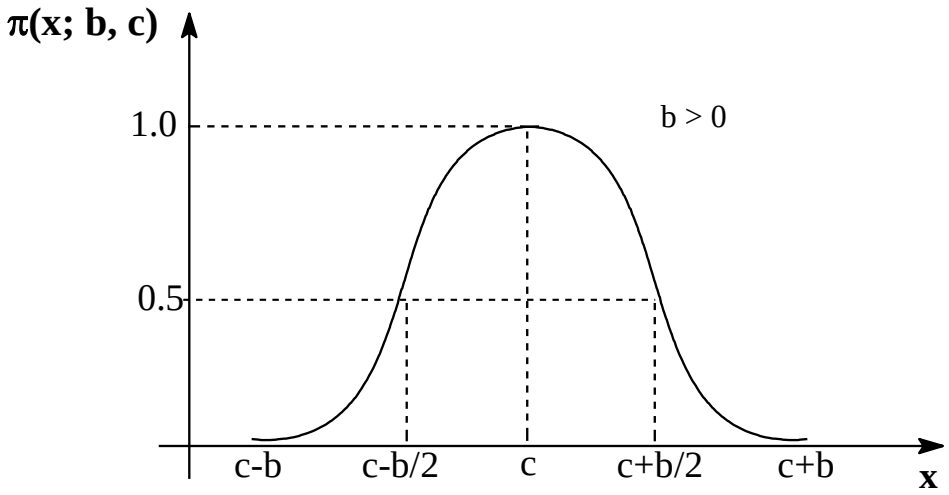


Fig.(6) The π-function.

6. Admissible and Nonadmissible Membership Functions

The assignment of the membership function of a fuzzy set is subjective in nature and, in general, reflects the context in which the problem is viewed. Although, the assignment of the membership function of a fuzzy set $\tilde{A}$ is "subjective", it can not be assigned arbitrarily, as the following example illustrate:

Example (6.1):

In this example, we consider the universal set as the class of real numbers, and the fuzzy set $\tilde{A}$ of a all real numbers that are much greater than 1. The fuzzy subset $\tilde{A}$ of the universal set $\mathbb{R}$ may be defined mathematically using the membership function, such as:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-1}{x}, & \text{if} \quad x > 1 \\ 0 \quad , & \text{if} \quad x \le 1 \end{cases}$$

While the function:

$$\mu_{\tilde{A}}(x) = \begin{cases} e^{-(x-1)}, & \text{if} \quad x > 1 \\ 0 \quad\quad , & \text{if} \quad x \le 1 \end{cases}$$

is monotonically decreases as x increases, and:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1-e^{-1000(x-1)}, & \text{if} \quad x > 1 \\ 0 \quad\quad\quad\quad , & \text{if} \quad x \le 1 \end{cases}$$

that increases monotonically, but is approximately equal to 1 for x = 1.1, which is not much greater than 1, hence it is not an adequate characteristic function. Functions like those are called nonadmissible membership functions related to the fuzzy set $\tilde{A}$. The function $\mu_{\tilde{A}}(x)$ as defined in this example and other functions, such as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1-e^{-0.1(x-1)}, & \text{if} \quad x > 1 \\ 0 \quad\quad\quad\quad , & \text{if} \quad x \le 1 \end{cases}$$

or:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{1}{\cosh(x-1)}, & \text{if} \quad x > 1 \\ 0 \qquad\qquad\quad , & \text{if} \quad x \leq 1 \end{cases}$$

which satisfy the condition that $\mu_{\tilde{A}}(x) \in [0, 1], \forall\, x \in X$ and are consistent with the specification of the considered fuzzy set, will be an admissible membership functions for $\tilde{A}$.

Differentiation and Integration of Fuzzy Functions

1. The Extension Principle

As it is pointed previously, one of the most basic concepts of fuzzy set theory, which can be used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle.

The extension principle may be defined as follows (see Remark (2.4)(17)):

Definition (1.1):

Let X be a Cartesian product of universes $X_1, X_2, \ldots, X_r$ and $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_r$ be r-fuzzy sets in $X_1, X_2, \ldots, X_r$, respectively, f is a mapping from X to a universe Y, $= f(x_1, x_2, \ldots, x_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, x_2, \ldots, x_r), x_1, x_2, \ldots, x_r \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \underset{(x_1,x_2,\ldots,x_r)\in f^{-1}(y)}{\text{Sup}} \text{Min}\left\{\mu_{\tilde{A}_1}(x_1),\ldots,\mu_{\tilde{A}_r}(x_r)\right\}, & \text{if} \quad f^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

where f^{-1} is the inverse operator of f.

For r = 1, the fuzzy extension principle, of course reduce to:

$$\tilde{B} = f(\tilde{A}) = f\{(y, \mu_{\tilde{B}}(y)) \mid y = f(x), x \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \underset{x\in f^{-1}(y)}{\text{Sup}} \mu_{\tilde{A}}(x), & \text{if} \quad f^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

Example (1.2):

Let $\tilde{A}$ ={(−2, 0.1), (0, 0.4), (1, 1), (2, 0.8)} and f(x) = x^2. Then by applying the extension principle, we obtain:

f(−2) = $(-2)^2$ = 4, with membership value 0.1

f(0) = $(0)^2$ = 0, with membership value 0.4

f(1) = $(1)^2$ = 1, with membership value 1

f(2) = $(2)^2$ = 4, with membership value 0.8

$\tilde{B}$ = {(0, 0.4), (1, 1), (4, max{0.1,0.8})}

= {(0, 0.4), (1,1), (4, 0.8)}

1.1 Fuzzifying Function:

Fuzzifying function from X to Y is the ordinary mapping of X into fuzzy power set $\tilde{P}(Y)$, $\tilde{f} : X \longrightarrow \tilde{f}(x)$, that is to say, the fuzzifying function is a mapping from domain to fuzzy set of range, fuzzifying function and the fuzzy relation coincides with each in the mathematical manner. So to speak, fuzzifying function can be interpreted as fuzzy relation $\tilde{R}$ defined as follows:

$\forall (x, y) \in X \times Y$, $\mu_{f(x)}(y) = \mu_{\tilde{R}}(x, y)$, where y = f(x).

Example (1.3):

Consider two crisp sets A = {2, 3, 4} and B = {2, 3, 4, 6, 8, 9, 12}. A fuzzifying function maps the elements in A to the power set P(B) in the following manner:

f(2) = $\tilde{B}_1$, f(3) = $\tilde{B}_2$, f(4) = $\tilde{B}_3$

here $\tilde{P}(B) = \{\tilde{B}_1, \tilde{B}_2, \tilde{B}_3\}$, and

$\tilde{B}_1$ = {(2, 0.5), (4, 1), (6, 0.5)}

$\tilde{B}_2$ = {(3, 0.5), (6, 1), (9, 0.5)}

$\tilde{B}_3$ = {(4, 0.5), (8, 1), (12, 0.5)}

The function f(x) maps element $2 \in A$ to element $2 \in B$, with degree 0.5 to element $4 \in B$, with membership 1 and to element $6 \in B$, with 0.5.

Now, we apply the α-cuts operation to the fuzzifying function:

$\tilde{f} : 2 \longrightarrow \{2, 4, 6\}$, for $\alpha = 0.5$

$\tilde{f} : 2 \longrightarrow \{4\}$, for $\alpha = 1.0$

In the same manner, we have:

$\tilde{f} : 3 \longrightarrow \{3, 6, 9\}$, for $\alpha = 0.5$

$\tilde{f} : 3 \longrightarrow \{6\}$, for $\alpha = 1.0$

Again:

$\tilde{f} : 4 \longrightarrow \{4, 8, 12\}$, for $\alpha = 0.5$

$\tilde{f} : 4 \longrightarrow \{8\}$, for $\alpha = 1.0$

Definition (1.5):

Let f be a fuzzifying function from $[a, b] \subseteq R$ to R, such that $\forall\, x \in [a, b]$, $\tilde{f}(x)$ is a fuzzy number, i.e., a piecewise continuous convex normalized fuzzy set on R. For all $\alpha \in (0, 1]$, the equation $\mu_{\tilde{f}(x)}(y) = \alpha$, with x and α as parameters is assumed to have two and only two continuous solutions $y = f_\alpha^+(x)$ and $y = f_\alpha^-(x)$, for $\alpha \neq 1$ and only one solution y = f(x), for $\alpha = 1$, which is also continuous. The functions f_α^+ and f_α^- are defined such that:

$f_{\alpha'}^+(x) \geq f_\alpha^+(x) \geq f(x) \geq f_\alpha^-(x) \geq f_{\alpha'}^-(x)$, $\forall\, \alpha, \alpha'$ with $\alpha' \leq \alpha$

These functions are called α-level curves of $\tilde{f}$.

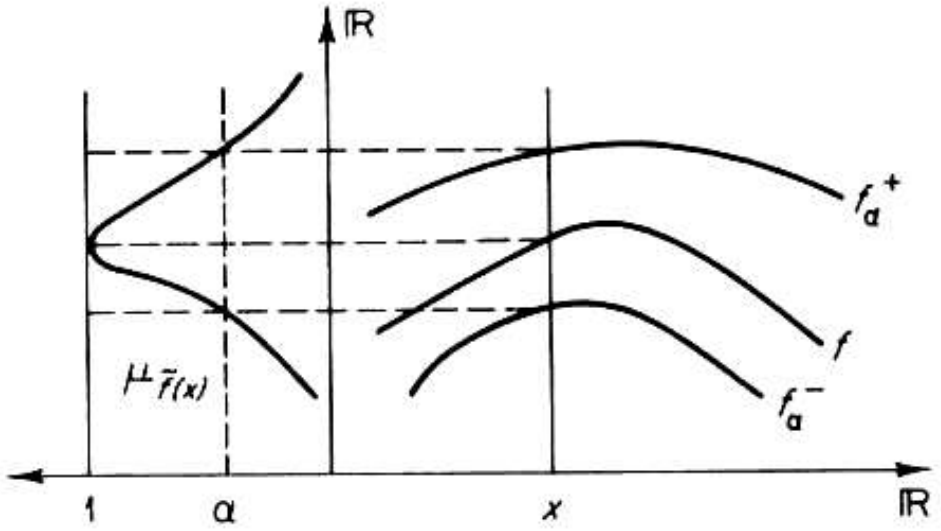


Fig.(1) α-Level curve of fuzzifying function.

1.2 LR-Type Representation of Fuzzy Number:

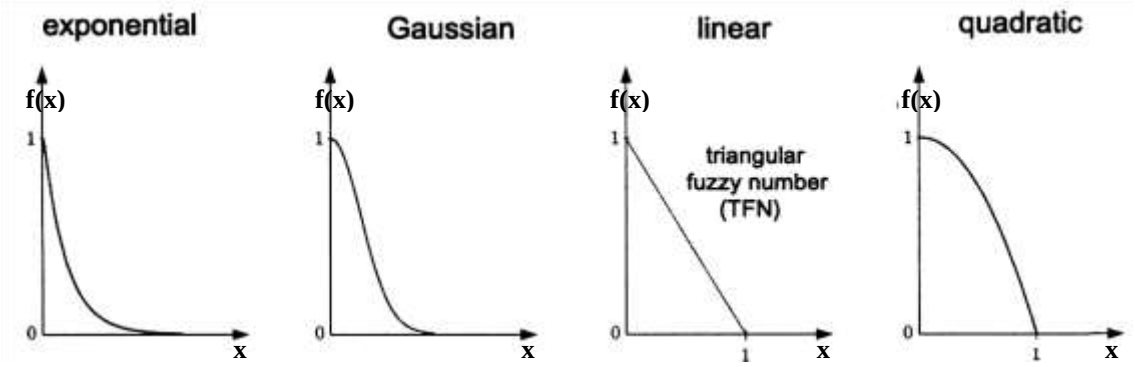
Dubois and Prade in 1980 suggested a special type of representation of fuzzy number of the following type: They are called L (and R), which map $R^+ \longrightarrow [0, 1]$ and are decreasing shape functions if $L(0) = 1$, $L(x) < 1$ for all $x > 0$; $L(x) > 0$ for all $x < 1$; $L(1) = 0$ or $[L(x) > 0$, for all x and $L(+\infty) = 0]$.

Definition (1.6):

A function f(x) is called the reference function of fuzzy number if and only if:

1. $f(-x) = f(x)$ (even function).
2. $f(0) = 1$.
3. f(x) is decreasing on $[0, +\infty)$.

For example, $f(x) = e^{-x}$, $f(x) = e^{-x^2}$, $f(x) = \max\{0, 1 - x^2\}$ and $f(x) = \max\{0, 1 - x\}$ are of such type of functions.



$f(x) = e^{-x}$ $f(x) = e^{-x^2}$ $f(x) = max\{0,1-x\}$ $f(x) = max\{0, 1-x^2\}$

Fig.(2).

Definition (1.7):

The fuzzy number $\tilde{M}$ is of LR-type if there exists reference functions, L (for left) and R (for right), and scalars $\alpha > 0$, $\beta > 0$, with:

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right), & \text{for } x \le m \\ R\left(\dfrac{x-m}{\beta}\right), & \text{for } x \ge m \end{cases}$$

m is called the mean value of $\tilde{M}$ it is a real number, and α and β are called the left and right spreads respectively. Symbolically $\tilde{M}$ is denoted by $(m, \alpha, \beta)_{LR}$.

Remarks (1.8):

1. When the spreads are zero, M is a nonfuzzy number by convection.
2. As the spreads increase $\tilde{M}$ becomes fuzzier and fuzzier.

Example (1.9):

Let:

$$L(x) = \frac{1}{1+x^2}$$

$$R(x) = \frac{1}{1+2|x|}$$

$\alpha = 2$, $\beta = 3$ and $m = 5$. Then:

$$\mu_{\tilde{5}}(x) = \begin{cases} L\left(\dfrac{5-x}{2}\right) & \text{for } x \le 5 \\ R\left(\dfrac{x-5}{3}\right), & \text{for } x \ge 5 \end{cases}$$

$$= \begin{cases} \dfrac{1}{1+\left(\dfrac{5-x}{2}\right)^2} & \text{for } x \le 5 \\ \dfrac{1}{1+\left|\dfrac{2(x-5)}{3}\right|}, & \text{for } x \ge 5 \end{cases}$$

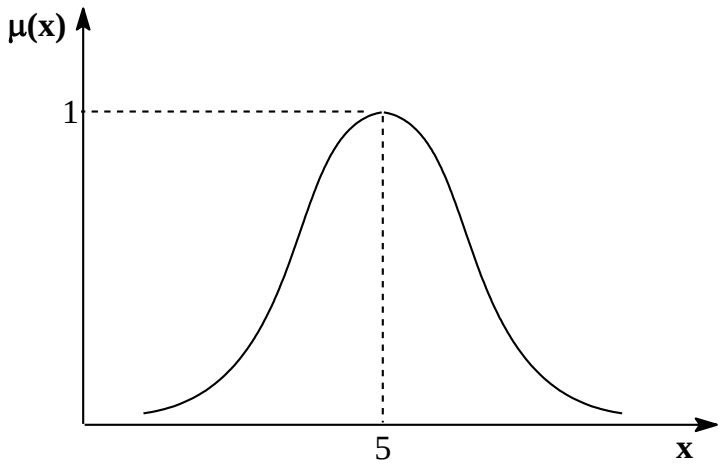


Fig.(3) LR-Representation of fuzzy number $\tilde{5}$.

2. Differentiation

Here, we will study two types of differentiations, which are the differentiation of nonfuzzy function defined over a fuzzy interval and the differentiation of a fuzzifying function over a nonfuzzy set.

2.1 Differentiation of Nonfuzzy (Crisp) Function Over Fuzzy Set:

By the extension principle, the differentiation $f'(\tilde{A})$ of a nonfuzzy function f at fuzzy point $\tilde{x}_0$ (A fuzzy point $\tilde{x}_0$ is convex fuzzy subset of the real line R) or fuzzy set $\tilde{A}$ is defined as:

$$\mu_{f'(\tilde{A})}(y) = \sup_{f'(x)=y} \mu_{\tilde{A}}(x)$$

Example (2.1):

Let $f(x) = x^3$ and the fuzzy point is given by:

$\tilde{A} = \{(-1, 0.4), (0, 1), (1, 0.6)\}$

then $f'(x) = 3x^2$ and therefore:

$f'(-1) = 3(-1)^2 = 3$, with membership 0.4

$f'(0) = 3(0)^2 = 0$, with membership 1

$f'(1) = 3(1)^2 = 3$, with membership 0.6

Hence:

$f'(\tilde{A}) = \{(3, 0.4), (0, 1), (3, 0.6)$

$= \{(3, 0.6), ((0, 1)\}$

2.2 Differentiation of Fuzzifying Function Over Nonfuzzy Set:

For all x in a domain D, the differentiation of fuzzifying function $\tilde{f}$ is define as follows; Let any α-cut of $\tilde{f}$ be differentiable for arbitrary x in D, we define differentiation $(d\tilde{f}/dx)(x_0)$ of $\tilde{f}$ at real point x_0 as:

$$\mu_{(d\tilde{f}/dx)(x_0)}(p) = \underset{f_\alpha:(df_\alpha/dx)(x_0)=p}{\text{Max}} \mu_{\tilde{f}}(f_\alpha)$$

Example (2.2):

Consider the fuzzifying function:

$\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)\}$

$f_1(x) = x$, $f_2(x) = x^2$ and $f_3(x) = x^3 + 1$

Then $f'_1(x) = 1$, $f'_2(x) = 2x$, $f'_3(x) = 3x^2$. Differentiation of this fuzzifying function at $x_0 = 0.5$ is obtained as:

$f'_1(0.5) = 1$, when $\alpha = 0.4$

$f'_2(0.5) = 1$, when $\alpha = 0.7$

$f'_3(0.5) = 3(0.5)^2 = 0.75$, when $\alpha = 0.4$.

Hence:

$$\frac{d\tilde{f}}{dx}(x_0) = \{(1, 0.4), (1, 0.7), (0.75, 0.4)\}$$

$$= \{(1, 0.7), (0.75, 0.4)\}$$

2.3 Differentiation of LR Type Fuzzy Functions:

Suppose that $\tilde{f}(x)$ is for any $x \in [a, b]$ and LR-fuzzy number. It means that for all $x \in [a, b]$:

$$\mu_{\tilde{f}(x)}(y) = \begin{cases} L\left(\dfrac{\psi(x) - y}{\lambda(x)}\right), & \text{if} \quad y \le \psi(x) \\ R\left(\dfrac{y - \psi(x)}{\rho(x)}\right), & \text{if} \quad y > \psi(x) \end{cases} \qquad \ldots(1)$$

or symbolically:

$$\tilde{f}(x) = (\psi(x), \lambda(x), \rho(x))_{LR}$$

where L and R are continuous strictly decreasing mappings from $[0, \infty)$ to $[0, 1]$ such that $L(0) = 1 = R(0)$ and $L(1) = 0 = R(1)$, where ψ is a differentiable mapping from $[a, b]$ to R, λ and ρ are positive differentiable mappings from $[a, b]$ to R+. $\psi(x)$ has a degree of membership to $\tilde{f}(x)$ equals to 1, $\lambda(x)$ and $\rho(x)$ are the left and right spreads of fuzzy number $\tilde{f}(x)$. Suppose that the reference functions are the same for any x, only their parameters depend on x, i.e., $\tilde{f}(x) = (\psi, \lambda, \rho)_{LR}$, then the α-curves have the following equations:

$$\left.\begin{aligned} f_\alpha^-(x) &= \psi(x) - L^{-1}(\alpha)\lambda(x) \\ f_1(x) &= \psi(x) \\ f_\alpha^+(x) &= \psi(x) + R^{-1}(\alpha)\rho(x) \end{aligned}\right\} \qquad \ldots(2)$$

after differentiation, we get:

$$\left.\begin{aligned} f_\alpha'^-(x_0) &= \psi'(x_0) - L^{-1}(\alpha)\lambda'(x_0) \\ f_1'(x_0) &= \psi'(x_0) \\ f_\alpha'^+(x_0) &= \psi'(x_0) + R^{-1}(\alpha)\rho'(x_0) \end{aligned}\right\} \qquad \ldots(3)$$

In eq.(2), $\forall\, x \in [a, b]$, $\lambda(x) > 0$ and $\rho(x) > 0$. It is no more necessarily that the case in eq.(3) for $\lambda'(x_0)$ and $\rho'(x_0)$. Therefore, we have to discuss according to the signs of $\lambda'(x_0)$ and $\rho'(x_0)$, such that:

1. $\lambda'(x_0) \geq 0$, $\rho'(x_0) \geq 0$. The α-section curves get a way from $\psi(x)$ when x increases, we have if $\beta < \alpha < 1$ (see Fig.(4.a))

$$f_{\beta}^{+\prime}(x_0) > f_{\alpha}^{+\prime}(x_0) > \psi'(x_0) > f_{\alpha}^{-\prime}(x_0) > f_{\beta}^{-\prime}(x_0)$$

there exists at most one level curve whose derivative in x_0 is equal to y:

$$\mu_{\tilde{f}'(x_0)}(y) = \begin{cases} L\left(\dfrac{\psi'(x_0) - y}{\lambda'(x_0)}\right), & \text{if} \quad y \leq \psi'(x_0) \\ R\left(\dfrac{y - \psi'(x_0)}{\rho'(x_0)}\right), & \text{if} \quad y > \psi'(x_0) \end{cases}$$

or:

$$f'(x_0) = (\psi'(x_0), \lambda'(x_0), \rho'(x_0))_{LR}$$

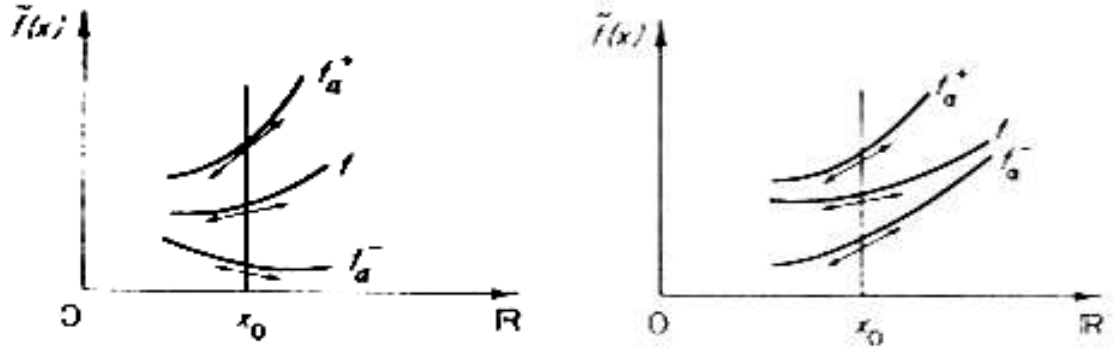
2. If $\lambda'(x_0) \leq 0$ and $\rho'(x_0) \leq 0$, this is the opposite case, $\lambda(x)$ and $\rho(x)$ are decreasing functions in a neighborhood of x_0, and if $\beta < \alpha < 1$, then:

$$f_{\beta}^{-\prime}(x_0) > f_{\alpha}^{-\prime}(x_0) > \psi'(x_0) > f_{\alpha}^{+\prime}(x_0) > f_{\beta}^{+\prime}(x_0)$$

or

$$\tilde{f}'(x_0) = (\psi'(x_0), -\rho'(x_0), -\lambda'(x_0))_{RL}$$

3. If $\lambda'(x_0) \leq 0$ and $\rho'(x_0) \geq 0$, then the bundle of α-section curves has a shape which is pictured in Fig.(2.4.b)



(a) ***(b)***

Fig.(4).

Now, $\forall\ \alpha$, Min $\left\{ f_{\alpha}^{+\prime}(x_0), f_{\alpha}^{-\prime}(x_0) \right\} \geq \psi'(x_0)$, thus:

$$\mu_{\tilde{f}'(x_0)}(y) = \begin{cases} \text{Max}\left\{ L\left(\dfrac{y-\psi'(x_0)}{-\lambda'(x_0)} \right), R\left(\dfrac{y-\psi'(x_0)}{\rho'(x_0)} \right) \right\}, & \text{if} \quad y \geq \psi'(x_0) \\ 0, & \text{if} \quad y < \psi'(x_0) \end{cases}$$

or:

$$\tilde{f}'(x_0) = (\psi'(x_0), 0, -\lambda'(x_0))_L \cup (\psi'(x_0), 0, \rho'(x_0))_R$$

4. If $\lambda'(x_0) \geq 0$ and $\rho'(x_0) \leq 0$, then this is an opposite of the preceding, and we have:

$$\tilde{f}'(x_0) = (\psi'(x_0), \lambda'(x_0), 0)_L \cup (\psi'(x_0), -\rho'(x_0), 0)_R$$

Example (2.3):

Let:

$$f(x) = x^2,\ s(x) = x^2/8,\ t(x) = x^2/4,\ L(x) = \frac{1}{1+x^2},\ R(x) = \frac{1}{1+2x}$$

Then:

$$s'(x) = x/4,\ t'(x) = x/2, f'(x) = 2x$$

Now, let $x_0 = 2$, $p = 2$, then $f'(x_0) = 4$, $s'(x_0) = ½$, $t'(x_0) = 1$, and since $2 \leq 4$, then:

$$\mu_{(df'/dx)(2)}(2) = L\left(\frac{4-2}{0.5} \right) = \frac{1}{17}$$

and therefore:

$$\frac{d\tilde{f}}{dx}(2) = (4, ½, 1)_{LR}$$

Now, let $x_0 = 4$, $p = 10$, then similarly $\dfrac{d\tilde{f}}{dx}(4) = (8, 1, 2)_{LR}$ and if $x_0 = 2$, $p = 2$, but $s(x) = -x^2/8$, $t(x) = -x^2/4$, hence:

$$\mu_{(df'/dx)(2)}(4) = L\left(\frac{4-2}{-0.5} \right) = \frac{1}{17}$$

and

$$\frac{d\tilde{f}}{dx}(2) = (4, 1, ½)_{RL}$$

If $x_0 = 1$, $p = 4$, then:

$$\mu_{(df'/dx)(1)}(4) = R\left(\frac{4-2}{-0.5}\right) = -\frac{1}{7}$$

and

$$\frac{d\tilde{f}}{dx}(1) = (2, ½, 0.25)_{RL}$$

If $s(x) = x^2/8$, $t(x) = -x^2/2$, $f(x) = x^2$

and if $p = 2$, then:

$\mu_{(df'/dx)(2)}(2) = 0.$

$$\frac{d\tilde{f}}{dx}(2) = (4, 0, ½)_L \cup (4, 0, 2)_R$$

3. Integration of a (Crisp) Real Valued Functin Over a Fuzzy Interval

A fuzzy domain $\tilde{D}$ of the real line R is assumed to be bounded by two normalized convex fuzzy sets. The membership function of each are $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively. $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ can be interpreted as the degree (of confidence) to which x can be considered a lower or upper bounds of $\tilde{D}$. If $\underline{a}_0$ and $\bar{b}_0$ are the lower/upper limits of the supports of $\tilde{a}$ or $\tilde{b}$, then a_0 or b_0 are related to each other by $\underline{a}_0 = \text{Inf } S(\tilde{a}) \le \text{Sup } S(\tilde{b}) = \bar{b}_0$.

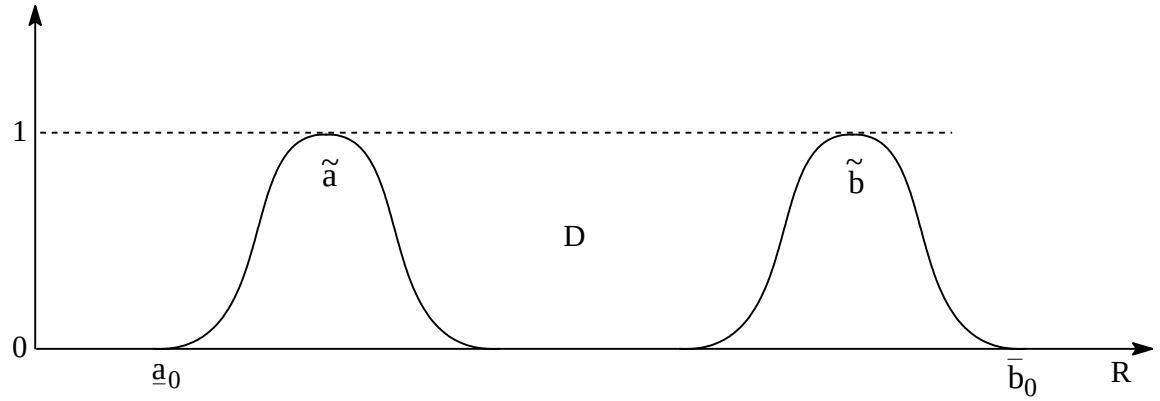


Fig.(5) Fuzzily bounded interval.

Definition (3.1):

Let f be a real valued function which is integrable in the interval $J = [a_0, b_0]$, then according to the extension principle the membership function of the fuzzy integral $\int_{\tilde{D}} f$ is given by:

$$\mu_{\int_{\tilde{D}} f}(z) = \underset{\substack{x,y\in J \\ z=\int_x^y f}}{\text{Sup}} \text{Min}\left\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\right\}$$

Example (3.2):

Let:

$\tilde{a} = \{(4, 0.8), (5, 1), (6, 0.4)\}$

$\tilde{b} = \{(6, 0.7), (7, 1), (8, 0.2)\}$

and

$f(x) = 2,\ x \in [a_0, b_0] = [4, 8]$

and the problem is to find the fuzzy integration of f(x) over J = [4, 8], then:

(a, b)	$\int_a^b 2\,dx = 2b - 2a$	$Min\{\mu_{\tilde{a}}(a),\ \mu_{\tilde{b}}(b)\}$
(4, 6)	4	0.7
(4, 7)	6	0.8
(4, 8)	8	0.2
(5, 6)	2	0.7
(5, 7)	4	1.0
(5, 8)	6	0.2
(6, 6)	0	0.4
(6, 7)	2	0.4
(6, 8)	4	0.2

and by using the definition (3.1), then:

$$\int_{\tilde{D}} f = \{(0, 0.4), (2, 0.7), (4, 1), (6, 0.8), (8, 0.2)\}$$

3. Solution of Fuzzy Differential Equations

3.1 Analytic Solution of Linear Fuzzy Differential Equations

Consider the system:

$$\tilde{y}'(x) = A\,\tilde{y}(x),\ \tilde{y}(0) = \tilde{y}_0 \qquad \ldots(3.1)$$

where $\tilde{y}'(x) = \dfrac{d\tilde{y}}{dx}$, $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\tilde{y}$ is a fuzzy mapping $\tilde{y} : \mathbb{R}^n \longrightarrow [0, 1]$, where $\tilde{y}$ is a vector made up of n-fuzzy mapping.

Each element of the vector $\tilde{y}$ in (3.1) at the intervals x written as:

$$\tilde{y}_r^k(x) = [\underline{\tilde{y}}_r^k(x),\ \overline{\tilde{y}}_r^k(x)],\ k = 1, 2, \ldots, n \qquad \ldots(3.2)$$

It is shown that the evaluation of the system (3.1) can be described by 2n differential equations for the end points of the variables (3.2), this for each given x and value of r of course, the complete overview have to be built up numerically by interpolation or some other means. The equation for the end points of the intervals are given by:

$$\left.\begin{aligned} \underline{\tilde{y}}_r'^k(x) &= \min\left\{(A\tilde{u})_k : \tilde{u}^i \in [\underline{\tilde{y}}_r^i(x), \overline{\tilde{y}}_r^i(x)]\right\} \\ \overline{\tilde{y}}_r'^k(x) &= \max\left\{(A\tilde{u})_k : \tilde{u}^i \in [\underline{\tilde{y}}_r^i(x), \overline{\tilde{y}}_r^i(x)]\right\} \\ \underline{\tilde{y}}_r^k(0) &= \underline{\tilde{y}}_{r_0}^k,\ \overline{\tilde{y}}_r^k(0) = \overline{\tilde{y}}_{r_0}^k \end{aligned}\right\} \qquad \ldots(3.3)$$

where $(A\tilde{u})_k := \sum_{j=1}^{n} a_{kj}\tilde{u}^j$ is the kth-row of $A\tilde{u}$.

The vector field in (3.1) is linear, and so the following rule applies in (3.3):

$$\underline{\tilde{y}}_r'^k(x) = \sum_{j=1}^{n} a_{kj}\tilde{u}^j \qquad \ldots(3.4)$$

where:

$$\tilde{u}^j = \begin{cases} \underline{\tilde{y}}_r^j(x), & \text{if} \quad a_{kj} \geq 0 \\ \overline{\tilde{y}}_r^j(x), & \text{if} \quad a_{kj} < 0 \end{cases}$$

and

$$\bar{\tilde{y}}_r'^k(x) = \sum_{j=1}^{n} a_{kj}\tilde{u}^j \qquad \ldots(3.5)$$

where:

$$\tilde{u}^j = \begin{cases} \bar{\tilde{y}}_r^j(x), & \text{if} \quad a_{kj} \geq 0 \\ \underline{\tilde{y}}_r^j(x), & \text{if} \quad a_{kj} < 0 \end{cases}$$

Equations (3.4) and (3.5) are called the parametric equations.

3.1.1 Representation by complex numbers:

This approach is based on the following criteria, for each variable in (3.1), therefore, there are two equations of the type (3.4) and (3.5), which could be written out explicitly.

However, we propose a slightly more compact representation of the same thing by passing to the field of complex numbers.

Define new variables n complex form:

$$\tilde{z}_r^k = \underline{\tilde{y}}_r^k + i\,\bar{\tilde{y}}_r^k \qquad \ldots(3.6)$$

where $i := \sqrt{-1}$ and the two operations carried on the complex numbers as:

$$e\,\tilde{z}_r^k = \tilde{z}_r^k$$

$$g(\tilde{z}_r^k) = \bar{\tilde{y}}_r^k + i\,\underline{\tilde{y}}_r^k \qquad \ldots(3.7)$$

e is just the identity operation and g corresponds to a flip about the diagonal in the complex plane. We notice that $g^2 = e$ and $g^k = e$ if k is even and $g^k = g$ if k is odd. It is easily verified that:

$$(\eta g)\tilde{z}_r^k = (g\eta)\tilde{z}_r^k \text{ for } \eta \in \mathbb{R}$$

and we extend g to vectors via:

$$g\,\tilde{z}_r = \begin{bmatrix} g\tilde{z}_r^1 \\ g\tilde{z}_r^2 \\ \vdots \\ g\tilde{z}_r^n \end{bmatrix}$$

Using (3.6) and the two operators (3.7), it is fairly easy to see that equations (3.4) and (3.5) can be written as:

$$\tilde{z}_r' = B\tilde{z}_r,\ \tilde{z}_r^k(0) = \tilde{z}_{r_0} \qquad \ldots(3.8)$$

where the elements of the matrix B are determined from those of A as follows:

$$b_{ij} = \begin{cases} ea_{ij}, & a_{ij} \geq 0 \\ ga_{ij}, & a_{ij} < 0 \end{cases} \qquad \ldots(3.9)$$

The solution of (3.8) is then given by:

$$\tilde{z}_r(x) = \exp(xB)\ \tilde{z}_{r_0} \qquad \ldots(3.10)$$

The problem is therefore to calculate the exponential of the matrix B, where certain elements are multiplied by the flip operator (3.9). This can be done for small values of x by first of all writing the matrix B as the sum of two matrices, one of which is multiplied by the operator e, and the other by g, for example:

$$B = eC + gD$$

Now, for small x, we have:

$$\exp(xB)\tilde{z}_{r_0} = \exp(x(eC + gD))\tilde{z}_{r_0}$$

$$= \exp(xeC)\exp(xgD)\tilde{z}_{r_0} + O(x) \qquad \ldots(3.11)$$

where O(x) is a function of x, i.e., $\lim_{x\to 0} O(x)/x = 0$.

The first part on the right hand side exp(xeC) is simply the standard matrix exponential, because e is the identity operator. For the second part exp(xgD), we note that $g^k = e$ if k is even and $g^k = g$ if it is odd and then proceed to calculate the formal series of exp (xgD):

$$\exp(xgD)\tilde{z}_{r_0} = \left(I + xgD + \frac{x^2}{2!}D^2 + \frac{x^3}{3!}gD^3 + \ldots\right)\tilde{z}_{r_0}$$

$$= \left(I + \frac{x^2}{2!}D^2 + \ldots\right)\tilde{z}_{r_0} + \left(xD + \frac{x^3}{3!}D^3 + \ldots\right)g\tilde{z}_{r_0}$$

$$= \cosh(xD)\tilde{z}_{r_0} + \sinh(xD)g\tilde{z}_{r_0} \qquad \ldots(3.12)$$

Combining (3.11) with (3.12), the solution (3.10) for small x is then given by:

$$\tilde{z}_r(x) = \exp(xC)(\cosh(xD)\ \tilde{z}_{r_0} + \sinh(xD)g\,\tilde{z}_{r_0})$$

letting:

$$\varphi(x) = \exp(xC)\cosh(xD) \qquad \ldots(3.13)$$

$$\psi(x) = \exp(xC)\sinh(xD) \qquad \ldots(3.14)$$

and so in component form, one have:

$$\tilde{z}_r(x) = \sum_{j=1}^{n} [\varphi_{kj}(x)\tilde{z}^j_{r_0} + \psi_{kj}(x)g\,\tilde{z}^j_{r_0}]$$

Then by (3.6) this reduces to:

$$\underset{\sim}{\tilde{y}}{}^k_r(x) + i\,\overline{\tilde{y}}{}^k_r(x) = \sum_{j=1}^{n} [\varphi_{kj}(x)(\underset{\sim}{\tilde{y}}{}^j_{r_0}(x) + i\,\overline{\tilde{y}}{}^j_{r_0}(x)) + \psi_{kj}(x)(\overline{\tilde{y}}{}^j_{r_0}(x) + i\,\underset{\sim}{\tilde{y}}{}^j_{r_0}(x))]$$

and in other words:

$$\underset{\sim}{\tilde{y}}{}^k_r(x) = \sum_{j=1}^{n} [\varphi_{kj}(x)\,\underset{\sim}{\tilde{y}}{}^j_{r_0}(x) + \psi_{kj}(x)\,\overline{\tilde{y}}{}^j_{r_0}(x)] \qquad \ldots(3.15)$$

$$\overline{\tilde{y}}{}^k_r(x) = \sum_{j=1}^{n} [\varphi_{kj}(x)\,\overline{\tilde{y}}{}^j_{r_0}(x) + \psi_{kj}(x)i\,\underset{\sim}{\tilde{y}}{}^j_{r_0}(x)] \qquad \ldots(3.16)$$

Example (3.1):

Consider the linear system $\tilde{y}' = A\tilde{y}$, where $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ with initial values to be $\tilde{y}^1(0)$ about 1 and $\tilde{y}^2(0)$ about -1, which are fuzzy numbers and using membership function defined by setting, for example,

$$\mu_{\tilde{y}^1_0}(x) = \begin{cases} 0, & x < 0 \\ 2x - x^2, & 0 \le x < 2 \\ 0, & x > 2 \end{cases}$$

and

$$\mu_{\tilde{y}_0^2}(x) = \begin{cases} 0, & x > -2 \\ -2x - x^2, & -2 \le x < 0 \\ 0, & x > 0 \end{cases}$$

Thus, for $r \in [0,1]$, we can represent the initial condition as:

$$[\tilde{y}_0^1]_r = [\underline{\tilde{y}}_{0_r}^1, \overline{\tilde{y}}_{0_r}^1] = [1 - \sqrt{1-r}, 1 + \sqrt{1-r}]$$

$$[\tilde{y}_0^2]_r = [\underline{\tilde{y}}_{0_r}^2, \overline{\tilde{y}}_{0_r}^2] = [-1 - \sqrt{1-r}, -1 + \sqrt{1-r}]$$

Now, to solve the above system, first we use equations (3.9) to have:

$$B = \begin{bmatrix} g(-1) & e(1) \\ e(0) & g(-2) \end{bmatrix} = \begin{bmatrix} -i & 1 \\ 0 & -2i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & -2i \end{bmatrix}$$

$$= e\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + g\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= eC + gD$$

Using equations (3.13) and (3.14) at $x = 0.2$, we have:

$$\varphi(0.2) = \begin{bmatrix} 1.020066 & 0.216213 \\ 0 & 1.08166 \end{bmatrix}$$

$$\psi(0.2) = \begin{bmatrix} -0.201333 & -0.082133 \\ 0 & -0.410606 \end{bmatrix}$$

So, if we let for simplicity:

$a = 1 - \sqrt{1-r}$, $b = 1 + \sqrt{1-r}$, $c = -1 - \sqrt{1-r}$ and $d = -1 + \sqrt{1-r}$, where $r \in [0,1]$

Then using equations (3.15) and (3.16) to have the true solutions given by:

$$\underline{\tilde{y}}_r^1(0.2) = 1.020066a + 0.216231c - 0.201333b - 0.082133d$$

$$\overline{\tilde{y}}_r^1(0.2) = 1.020066b + 0.216213 - 0.201333a - 0.082133c$$

$$\underline{\tilde{y}}_r^2(0.2) = 1.0810666c - 0.410666d$$

$$\overline{\tilde{y}}_r^2(0.2) = -0.4106666c + 1.080666d$$

For example, if r = 0.1, then a = 0.0513167, b = 1.94867, c = –1.94868 and d = –0.051317.

$\underline{\tilde{y}}^{1}_{0.1}(0.2) = -0.7570682$

$\overline{\tilde{y}}^{1}_{0.1}(0.2) = 2.126394$

$\underline{\tilde{y}}^{2}_{0.1}(0.2) = -2.085526$

$\overline{\tilde{y}}^{2}_{0.1}(0.2) = 0.744773$

3.2 Linear Multistep Methods for Solving Fuzzy Differential Equations

Consider a first order fuzzy initial value problem, given by:

$$\left.\begin{aligned} \tilde{y}'(x) &= f(x, \tilde{y}(x)),\ x \in [x_0, b] \\ \tilde{y}(x_0) &= \tilde{y}_0 \end{aligned}\right\} \qquad \ldots(3.17)$$

where $\tilde{y}$ is a fuzzy function of x, f(x, $\tilde{y}(x)$) is a fuzzy function of the crisp variable x, and the fuzzy variable $\tilde{y}$ and $\tilde{y}'$ is the fuzzy derivative of $\tilde{y}$ and $\tilde{y}(x_0) = \tilde{y}_0$ is a triangular shaped fuzzy number.

The r-level set of $\tilde{y}(x)$ for $x \in [x_0, b]$ is $[\tilde{y}(x)]_r = [\underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)]$. Also $[\tilde{y}'(x)]_r = [\underline{\tilde{y}}'(x; r), \overline{\tilde{y}}'(x; r)]$, and

$$[f(x, \tilde{y}(x))]_r = [\underline{f}(x, \tilde{y}(x); r), \overline{f}(x, \tilde{y}(x); r)]$$

$$= [G(x, \underline{\tilde{y}}, \overline{\tilde{y}}), F((x, \underline{\tilde{y}}, \overline{\tilde{y}})]$$

Because of $\tilde{y}' = f(x, \tilde{y})$, we have:

$$\underline{\tilde{y}}'(x; r) = \underline{f}(x, \tilde{y}(x); r) = G(x, \underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)) \qquad \ldots(3.18)$$

$$\overline{\tilde{y}}'(x; r) = \overline{f}(x, \tilde{y}(x); r) = F(x, \underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)) \qquad \ldots(3.19)$$

Also:

$$[\tilde{y}(x_0)]_r = [\tilde{y}_0]_r = [\underline{\tilde{y}}_0(r), \overline{\tilde{y}}_0(r)]$$

By using the extension principle, we have the membership function:

$$f(x, \tilde{y}(x))(s) = \sup_{s \in R} \{ \mu_{\tilde{y}(x)}(\tau) \mid s = f(x, \tau) \} \quad \ldots(3.20)$$

Hence, it follows that:

$$[f(x, \tilde{y}(x))]_r = [\underline{f}(x, \tilde{y}(x); r), \overline{f}(x, \tilde{y}(x); r)], r \in [0, 1] \quad \ldots(3.21)$$

where:

$$\underline{f}(x, \tilde{y}(x); r) = \min \{f(x,u) \mid u \in [\tilde{y}(x)]_r\} \quad \ldots(3.22)$$

$$\overline{f}(x, \tilde{y}(x); r) = \max \{f(x,u) \mid u \in [\tilde{y}(x)]_r\} \quad \ldots(3.23)$$

In equation (3.17), we take f to be continuous function satisfying the Lipschitz condition which is the sufficient condition for the existence of a unique solution of equation (3.17).

Now, let $\tilde{Y} = [\underline{\tilde{Y}}, \overline{\tilde{Y}}]$ be the exact solution and $\tilde{y} = [\underline{\tilde{y}}, \overline{\tilde{y}}]$ be the approximate solution of the fuzzy initial value problem (3.17). Let:

$$[\tilde{Y}(x)]_r = [\underline{\tilde{Y}}(x; r), \overline{\tilde{Y}}(x; r)]$$

$$[\tilde{y}(x)]_r = [\underline{\tilde{y}}(x; r), \overline{\tilde{y}}(x; r)]$$

Also, the value of r throughout each step is unchanged. The exact and approximate solution at x_n denoted respectively for all n = 0, 1, …, N, by:

$$[\tilde{Y}_n(x)]_r = [\underline{\tilde{Y}}_n(x; r), \overline{\tilde{Y}}_n(x; r)]$$

$$[\tilde{y}_n(x)]_r = [\underline{\tilde{y}}_n(x; r), \overline{\tilde{y}}_n(x; r)]$$

The grid points at which the step of the solution is calculated through the step length $h = (x - x_0) / N$ and therefore, $x_i = x_0 + ih$, i = 0, 1, …, N. By using the general form of linear multistep method it is obtained that:

$$\underline{\tilde{Y}}_{n+k}(r) = h\sum_{j=0}^{k} \beta_j^* G(x_{n+j}, \underline{\tilde{Y}}_{n+j}(r), \overline{\tilde{Y}}_{n+j}(r)) - \sum_{j=0}^{k-1} \alpha_j^* \underline{\tilde{Y}}_{n+j}(r) + O(h^2) \quad \ldots(3.24)$$

and

$$\overline{\tilde{Y}}_{n+k}(r) = h\sum_{j=0}^{k} \beta_j F(x_{n+j}, \underline{\tilde{Y}}_{n+j}(r), \overline{\tilde{Y}}_{n+j}(r)) - \sum_{j=0}^{k-1} \alpha_j \overline{\tilde{Y}}_{n+j}(r) + O(h^2) \quad \ldots(3.25)$$

Omitting $O(h^2)$ in equations (3.24) and (3.25) will give the numerical solution as:

$$\underline{\tilde{y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j^{*}G(x_{n+j}, \underline{\tilde{y}}_{n+j}(r), \overline{\tilde{y}}_{n+j}(r)) - \sum_{j=0}^{k-1}\alpha_j^{*}\underline{\tilde{y}}_{n+j}(r) \quad \ldots(3.26)$$

and

$$\overline{\tilde{y}}_{n+k}(r) = h\sum_{j=0}^{k}\beta_j F(x_{n+j}, \underline{\tilde{y}}_{n+j}(r), \overline{\tilde{y}}_{n+j}(r)) - \sum_{j=0}^{k-1}\alpha_j\overline{\tilde{y}}_{n+j}(r) \quad \ldots(3.27)$$

3.3 Numerical Results

In this section, some numerical examples are presented as an illustration to the numerical methods discussed in the last section. For each example, numerical and theoretical results are presented in tables in order to give a comparison between the results.

3.3.1 Examples of Linear Fuzzy Differential Equations:

First, in this subsection, we will present some examples in linear case, i.e., examples of LFDE.

Example (3.2):

Consider the fuzzy initial value problem:

$\tilde{y}'(x) = \tilde{y}(x), \; x \in [0, 1]$

$\tilde{y}_r(0) = (0.75 + 0.25r, \; 1.125 - 0.125r), \; r \in [0, 1]$

The parametric form of $\tilde{y}(x)$ in this case is given by:

$$\underline{\tilde{y}}'(x; r) = \underline{\tilde{y}}(x; r) \;\text{ and }\; \overline{\tilde{y}}'(x; r) = \overline{\tilde{y}}(x; r) \quad \ldots(3.47)$$

with initial condition are given for all $r \in [0, 1]$ by:

$\underline{\tilde{y}}(0; r) = 0.75 + 0.25r$

$\overline{\tilde{y}}(0; r) = 1.125 - 0.125r$

The exact solution could be evaluated as:

$\underline{\tilde{Y}}_r(x) = e^{(x - x0)}(0.75 + 0.25r)$

$$\bar{\bar{Y}}_r(x) = e^{(x-x0)}(1.125 - 0.125r)$$

at $x = 1$ and $x_0 = 0$, we get:

$$\underline{\tilde{Y}}(1; r) = (0.75 + 0.25r)e$$

$$\bar{\bar{Y}}(1; r) = (1.125 - 0.125r)e$$

Using Euler method given by:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_n(r) + hG[x_n, \underline{\tilde{y}}_n(r), \bar{\bar{y}}_n(r)]$$

$$\bar{\bar{y}}_{n+1}(r) = \bar{\bar{y}}_n(r) + hF[x_n, \underline{\tilde{y}}_n(r), \bar{\bar{y}}_n(r)] \quad \ldots(3.48)$$

Using equations (3.47) with (3.48), we get:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_n(r) + h\underline{\tilde{y}}_n(r)$$

$$\bar{\bar{y}}_{n+1}(r) = \bar{\bar{y}}_n(r) + h\bar{\bar{y}}_n(r)$$

and Simpson method

$$\underline{\tilde{y}}_{n+2}(r) - \underline{\tilde{y}}_n(r) = \frac{h}{3}[G[x_{n+2}, \underline{\tilde{y}}_n(r) + 2hG(x_n, \underline{\tilde{y}}_n(r), \bar{\bar{y}}_n(r)), \bar{\bar{y}}_n(r) + 2hF(x_n, \underline{\tilde{y}}_n(r), \bar{\bar{y}}_n(r))] + 4G(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \bar{\bar{y}}_{n+1}(r)) + G(x_n, \underline{\tilde{y}}_n(r), \bar{\bar{y}}_n(r))]$$

$$\bar{\bar{y}}_{n+2}(r) - \bar{\bar{y}}_n(r) = \frac{h}{3}[F[x_{n+2}, \underline{\tilde{y}}_n(r) + 2hG(x_n, \underline{\tilde{y}}_n(r), \bar{\bar{y}}_n(r)), \bar{\bar{y}}_n(r) + 2hF(x_n, \underline{\tilde{y}}_n(r), \bar{\bar{y}}_n(r))] + 4F(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \bar{\bar{y}}_{n+1}(r)) + F(x_n, \underline{\tilde{y}}_n(r), \bar{\bar{y}}_n(r))] \ldots\ldots \ldots(3.49)$$

Using (3.47) with (3.49), to have:

$$\underline{\tilde{y}}_{n+2}(r) - \underline{\tilde{y}}_n(r) = \frac{h}{3}(\underline{\tilde{y}}_n(r) + 2h\underline{\tilde{y}}_n(r) + 4\underline{\tilde{y}}_{n+1}(r) + \underline{\tilde{y}}_n(r))$$

$$\bar{\bar{y}}_{n+2}(r) - \bar{\bar{y}}_n(r) = \frac{h}{3}(\bar{\bar{y}}_n(r) + 2h\bar{\bar{y}}_n(r) + 4\bar{\bar{y}}_{n+1}(r) + \bar{\bar{y}}_n(r))$$

with:

$$\underline{\tilde{y}}_1 = \underline{\tilde{y}}_0 + h\underline{\tilde{y}}_0 + \frac{h^2}{2}\underline{\tilde{y}}_0 \quad \text{and} \quad \bar{\bar{y}}_1 = \bar{\bar{y}}_0 + h\bar{\bar{y}}_0 + \frac{h^2}{2}\bar{\bar{y}}_0 \quad \text{(evaluated using Taylor method)}$$

as the initial value with $h = 0.1$.

The results are presented in table (3.1) after carrying out the computer programs.

Table (3.1) Results of example (3.2) with h = 0.1.

r	Euler results $(\underline{\tilde{y}}(r), \overline{\tilde{y}}(r))$	Simpson results $(\underline{\tilde{y}}(r), \overline{\tilde{y}}(r))$	Exact results $(\underline{\tilde{y}}(r), \overline{\tilde{y}}(r))$
0	(1.945307, 2.91796)	(2.036928, 3.055392)	(2.038711, 3.058067)
0.2	(2.074994, 2.853117)	(2.172723, 2.987494)	(2.174625, 2.99011)
0.4	(2.204681, 2.788273)	(2.308518, 2.919597)	(2.310539, 2.922153)
0.6	(2.334368, 2.72343)	(2.444314, 2.851699)	(2.446454, 2.854196)
0.8	(2.464055, 2.658586)	(2.580109, 2.783802)	(2.582368, 2.786239)
1.0	(2.593742, 2.593742)	(2.715904, 2.715904)	(2.718282, 2.718282

Example (3.3):

In order to solve the FIVP:

$$\tilde{y}'(x) = x\,\tilde{y}(x),\ x \in [-1, 1]$$

with initial condition in parametric form:

$$\tilde{y}(-1) = (\underline{\tilde{y}}_{r0},\ \overline{\tilde{y}}_{r0})$$
$$= (\sqrt{e} - 0.5(1 - r),\ \sqrt{e} + 0.5(1 - r))$$

The method of solution will be discussed for $x > 0$ and then for $x \le 0$ as the following cases show:

a.If x < 0: The parametric form in this case:

$$\underline{\tilde{y}}'(x; r) = x\,\overline{\tilde{y}}(x; r) \quad \text{and} \quad \overline{\tilde{y}}'(x; r) = x\,\underline{\tilde{y}}(x; r)$$

with initial conditions are:

$$\underline{\tilde{y}}(-1; r) = \underline{\tilde{y}}_{r0} \quad \text{and} \quad \overline{\tilde{y}}(-1; r) = \overline{\tilde{y}}_{r0}$$

The unique solution is given by:

$$\underline{\tilde{Y}}(x; r) = \cosh\left(\frac{x^2 - x_0^2}{2}\right)\underline{\tilde{y}}_{r0} + \sinh\left(\frac{x^2 - x_0^2}{2}\right)\overline{\tilde{y}}_{r0}$$
$$= \frac{A + B}{2}\,\underline{\tilde{y}}_{r0} + \frac{A - B}{2}\,\overline{\tilde{y}}_{r0}$$

$$\overline{\tilde{Y}}(x; r) = \cosh\left(\frac{x^2 - x_0^2}{2}\right)\overline{\tilde{y}}_{r0} + \sinh\left(\frac{x^2 - x_0^2}{2}\right)\underline{\tilde{y}}_{r0}$$

where $A = e^{\frac{x^2 - x_0^2}{2}}$, $B = \frac{1}{A}$

For $x = 0$, $x_0 = -1$, one get:

$$\underline{\tilde{Y}}(0; r) = \frac{e^{-1/2} + e^{1/2}}{2} \underline{\tilde{y}}_{r0} + \frac{e^{-1/2} - e^{1/2}}{2} \overline{\tilde{y}}_{r0} = \underline{\tilde{Y}}_{r0}$$

$$\overline{\tilde{Y}}(0; r) = \frac{e^{-1/2} + e^{1/2}}{2} \overline{\tilde{y}}_{r0} + \frac{e^{-1/2} - e^{1/2}}{2} \underline{\tilde{y}}_{r0} = \overline{\tilde{Y}}_{r0}$$

and using these numbers as initial conditions for the next step.

b.If $x \geq 0$: The parametric are given by:

$$\underline{\tilde{y}}'(x; r) = x\, \underline{\tilde{y}}(x; r) \quad \text{and} \quad \overline{\tilde{y}}'(x; r) = x\, \overline{\tilde{y}}(x; r)$$

with initial conditions are $\underline{\tilde{y}}_{r0}$, $\overline{\tilde{y}}_{r0}$.

The unique solution at $x \geq 0$ is:

$$\underline{\tilde{Y}}(x; r) = \underline{\tilde{Y}}_{r0}\, e^{\frac{x^2 - x_0^2}{2}}$$

$$\overline{\tilde{Y}}(x; r) = \overline{\tilde{Y}}_{r0}\, e^{\frac{x^2 - x_0^2}{2}}$$

Substituting the arguments of $\underline{\tilde{Y}}_{r0}$, $\overline{\tilde{Y}}_{r0}$ in the last result, these leading to:

$$\underline{\tilde{Y}}(1; r) = \frac{1 - e}{2} \overline{\tilde{y}}_{r0} + \frac{1 + e}{2} \underline{\tilde{y}}_{r0}$$

$$\overline{\tilde{Y}}(1; r) = \frac{1 + e}{2} \overline{\tilde{y}}_{r0} - \frac{e - 1}{2} \underline{\tilde{y}}_{r0}$$

To get method approximation, we divide $[-1,1]$ into (even number) N with equally spaced subintervals. In the numerical calculations, we use:

$$\underline{\tilde{y}}_1(r) = \underline{\tilde{y}}_0 + h x_0 \overline{\tilde{y}}_0 + \frac{h^2}{2}(1 + x_0^2)\, \underline{\tilde{y}}_0$$

$$\overline{\tilde{y}}_1(r) = \overline{\tilde{y}}_0 + h x_0 \underline{\tilde{y}}_0 + \frac{h^2}{2}(1 + x_0^2)\, \overline{\tilde{y}}_0$$

as initial value of two step method.

Now, upon carrying the midpoint method, such that:

$$\underline{\tilde{y}}_{n+2}(r) = \underline{\tilde{y}}_n(r) + 2hG(x_{n+1}, \underline{\tilde{y}}_{n+1}(r), \overline{\tilde{y}}_{n+1}(r))$$

$$\overline{\tilde{y}}_{n+2}(r) = \overline{\tilde{y}}_{n}(r) + 2hF(x_{n+1},\ \underline{\tilde{y}}_{n+1}(r),\ \overline{\tilde{y}}_{n+1}(r))$$

when $x_n \geq 0$. Then:

$$\underline{\tilde{y}}_{n+2}(r) = \underline{\tilde{y}}_{n}(r) + 2hx_{n+1}\,\underline{\tilde{y}}_{n+1}(r)$$

$$\overline{\tilde{y}}_{n+2}(r) = \overline{\tilde{y}}_{n}(r) + 2hx_{n+1}\,\overline{\tilde{y}}_{n+1}(r)$$

In case of $x_n < 0$. Then:

$$\underline{\tilde{y}}_{n+2}(r) = \underline{\tilde{y}}_{n}(r) + 2hx_{n+1}\,\overline{\tilde{y}}_{n+1}(r)$$

$$\overline{\tilde{y}}_{n+2}(r) = \overline{\tilde{y}}_{n}(r) + 2hx_{n+1}\,\underline{\tilde{y}}_{n+1}(r)$$

and trapezium method, given by:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_{n}(r) + \frac{h}{2}[G(x_{n+1},\ \underline{\tilde{y}}_{n+1}(r),\ \overline{\tilde{y}}_{n+1}(r)) + G(x_n,\ \underline{\tilde{y}}_{n}(r),\ \overline{\tilde{y}}_{n}(r))]$$

$$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_{n}(r) + \frac{h}{2}[F(x_{n+1},\ \underline{\tilde{y}}_{n+1}(r),\ \overline{\tilde{y}}_{n+1}(r)) + F(x_n,\ \underline{\tilde{y}}_{n}(r),\ \overline{\tilde{y}}_{n}(r))]$$

when $x_n \geq 0$:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_{n}(r) + \frac{h}{2}[x_{n+1}\,\underline{\tilde{y}}_{n+1}(r) + x_n\,\underline{\tilde{y}}_{n}(r)]$$

$$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_{n}(r) + \frac{h}{2}[x_{n+1}\,\overline{\tilde{y}}_{n+1}(r) + x_n\,\overline{\tilde{y}}_{n}(r)]$$

when $x_n < 0$, one have:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_{n}(r) + \frac{h}{2}[x_{n+1}\,\overline{\tilde{y}}_{n+1}(r) + x_n\,\overline{\tilde{y}}_{n}(r)]$$

$$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_{n}(r) + \frac{h}{2}[x_{n+1}\,\underline{\tilde{y}}_{n+1}(r) + x_n\,\underline{\tilde{y}}_{n}(r)]$$

The results are presented in table (3.2), when $x < 0$ and in table (3.3) when $x \geq 0$.

Table (3.2)Results of example (3.3) with h = 0.1, for $x_n < 0$.

r	Midpoint results	Trapezium results	Exact results
0	(0.1741345, 1.831501)	(0.4982622, 1.876977)	(0.1756394, 1.824361)
0.2	(0.3398712, 1.665765)	(0.6361336, 1.739105)	(0.3405115, 1.659489)
0.4	(0.5056078, 1.500028)	(0.7740051, 1.601234)	(0.5053836, 1.494616)
0.6	(0.6713445, 1.334291)	(0.9118765, 1.463362)	(0.6702557, 1.329744)
0.8	(0.8370812, 1.168555)	(1.049748, 1.325491)	(0.8351279, 1.164872)
1.0	(1.002818, 1.002818)	(1.187619, 1.187619)	(1, 1)

Table (3.3)Results of example (3.3) with h = 0.1, for $x_n \geq 0$

r	Midpoint results	Trapezium results	Exact results
0	(0.2627423, 2.729096)	(1.031197, 2.90632)	(0.2895803, 3.007862)
0.2	(0.5093776, 2.48246)	(1.218709, 2.718808)	(0.5614085, 2.736034)
0.4	(0.756013, 2.235825)	(1.406222, 2.531295)	(0.8332367, 2.464206)
0.6	(1.002648, 1.98919)	(1.593734, 2.343783)	(1.105065, 2.192378)
0.8	(1.249284, 1.742554)	(1.781246, 2.156271)	(1.376893, 1.92055)
1.0	(1.495919, 1.495919)	(1.968758, 1.968758)	(1.648721, 1.648721)

Example (3.4):

Consider the non-linear FIVP:

$$\tilde{y}'(x) = e^{-\tilde{y}^2(x)},\ y(0) = (0.75 + 0.25r,\ 1.5 - 0.5r)$$

over the interval [0,1].

The parametric equations are:

$$\underline{\tilde{y}}'(x; r) = \exp\{-[\max(|\underline{\tilde{y}}(x; r)|,\ |\overline{\tilde{y}}(x; r)|)]^2\}$$

$$\overline{\tilde{y}}'(x; r) = \exp\{-[\min(|\underline{\tilde{y}}(x; r)|,\ |\overline{\tilde{y}}(x; r)|)]^2\} \quad \ldots(3.69)$$

with the initial condition $(\underline{\tilde{y}}(0;r), \overline{\tilde{y}}(0;r))$. This example could not be solved analytically, therefore numerical methods will be used for different values of step size h.

Using Adam-Bashforth method:

$$\underline{\tilde{y}}_{n+3}(r) - \underline{\tilde{y}}_{n+2}(r) = \frac{h}{12}[23G(x_{n+2},\ \underline{\tilde{y}}_{n+2}(r),\ \overline{\tilde{y}}_{n+2}(r)) - 16G(x_{n+1},\ \underline{\tilde{y}}_{n+1}(r),\ \overline{\tilde{y}}_{n+1}(r)) + 5\ G(x_n,\ \underline{\tilde{y}}_n(r),\ \overline{\tilde{y}}_n(r))]$$

$$\overline{\tilde{y}}_{n+3}(r) - \overline{\tilde{y}}_{n+2}(r) = \frac{h}{12}[23F(x_{n+2},\ \underline{\tilde{y}}_{n+2}(r),\ \overline{\tilde{y}}_{n+2}(r)) - 16F(x_{n+1},\ \underline{\tilde{y}}_{n+1}(r),\ \overline{\tilde{y}}_{n+1}(r)) + 5F(x_n,\ \underline{\tilde{y}}_n(r),\ \overline{\tilde{y}}_n(r))] \quad \ldots(3.70)$$

Now, take equations (3.69) and (3.70), to have:

$$\underline{\tilde{y}}_{n+3}(r) = \underline{\tilde{y}}_{n+2}(r) + \frac{h}{12}[23\exp\{-[\max(|\underline{\tilde{y}}_{n+2}(x; r)|, |\overline{\tilde{y}}_{n+2}(x; r)|)]^2\} - 16\exp\{-[\max(|\underline{\tilde{y}}_{n+1}(x; r)|, |\overline{\tilde{y}}_{n+1}(x; r)|)]^2\} + 5\exp\{-[\max(|\underline{\tilde{y}}_n(x; r)|, |\overline{\tilde{y}}_n(x; r)|)]^2\}]$$

$$\overline{\tilde{y}}_{n+3}(r) = \overline{\tilde{y}}_{n+2}(r) + \frac{h}{12}[23\exp\{-[\min(|\underline{\tilde{y}}_{n+2}(x; r)|, |\overline{\tilde{y}}_{n+2}(x; r)|)]^2\} - 16\exp\{-[\min(|\underline{\tilde{y}}_{n+1}(x; r)|, |\overline{\tilde{y}}_{n+1}(x; r)|)]^2\} + 5\exp\{-[\min(|\underline{\tilde{y}}_n(x; r)|, |\overline{\tilde{y}}_n(x; r)|)]^2\}]$$

with using Eular method as an initial condition, such that:

$$\underline{\tilde{y}}_{n+1}(r) = \underline{\tilde{y}}_n(r) + h\exp\{-[\max(|\underline{\tilde{y}}_n(x; r)|, |\overline{\tilde{y}}_n(x; r)|)]^2\}$$

$$\overline{\tilde{y}}_{n+1}(r) = \overline{\tilde{y}}_n(r) + h\exp\{-[\min(|\underline{\tilde{y}}_n(x; r)|, |\overline{\tilde{y}}_n(x; r)|)]^2\}$$

for n = 0, 1.

The results are presented in table (3.5) using the computer programs.

Table (3.5) Numerical results of example (3.4) using Adam's method.

r	Adam-Bashforth results with h = 0.1		Adam-Bashforth results with h = 0.05	
	$\underline{\tilde{y}}(r)$	$\overline{\tilde{y}}(r)$	$\underline{\tilde{y}}(r)$	$\overline{\tilde{y}}(r)$
0.0	0.7962322	1.93692	0.7971708	1.989998
0.2	0.8666981	1.795828	0.8690056	1.842866
0.4	0.9441779	1.653868	0.9487784	1.694455
0.6	1.029957	1.511592	1.03812	1.545408
0.8	1.124981	1.369808	1.138298	1.396737
1.0	1.229553	1.229553	1.249807	1.249807

For simplicity, the following abbreviations are used for $I \equiv [0,1]$, $I^X \equiv$ the set of all fuzzy subsets of X, and $X^* \equiv$ the set of all closed and bounded fuzzy subsets o X.

The next two definitions are of great importanct in the present chapter.

Definition 3.1:

A is closed and bounded fuzzy susbet of X if and only if for all $\alpha \in I$, A_α is closed and bounded subset of X.

The distance function is given as in the following definition,

Definition 3.2:

Let (X,d) be a compact metric space, and $A, B \in X^*$ be any two closed and bounded fuzzy subset of X (i.e., A_α, B_α are closed and bounded subsets of X for each $\alpha \in I$). Define the fuzzy distance function $d^* : X^* \times X^* \longrightarrow \mathbb{R}^+$, by:

$$d^*(A,B) = \sup_{\alpha \in I}\left\{\max\left\{\sup_{x\in A_\alpha}\inf_{y\in B_\alpha} d(x,y), \sup_{y\in B_\alpha}\inf_{x\in A_\alpha} d(y,x)\right\}\right\}. \qquad (3.1)$$

In order to ensure the existence of infimum and supremum, it is necessary to restrict our attention to the set X^* of all non empty compact subsets of X.

In the present stage, we have to show that the function defined by (3.1) is really a distance function.

Theorem 3.1:

(X^*,d^*) is a metric space.

Proof:

We have to prove that the following three conditions are satisfied for all A, B and C $\in X^*$

1- $d^*(A,B) = d^*(B,A)$,

2- $d^*(A,B) \geq 0$ and $d^*(A,B) = 0$ if and only if $A = B$,

3- $d^*(A,B) \leq d^*(A,C) + d^*(C,B)$.

For the first condition,

$$d^*(A,B) = \sup_{\alpha \in I}\left\{\max\left\{\sup_{x\in A_\alpha}\inf_{y\in B_\alpha} d(x,y), \sup_{y\in B_\alpha}\inf_{x\in A_\alpha} d(y,x)\right\}\right\}$$

$$= \sup_{\alpha \in I}\left\{\max\left\{\sup_{x\in A_\alpha}\inf_{y\in B_\alpha} d(x,y), \sup_{y\in B_\alpha}\inf_{x\in A_\alpha} d(x,y)\right\}\right\}$$

$$= \sup_{\alpha \in I}\left\{\max\left\{\sup_{y\in B_\alpha}\inf_{x\in A_\alpha} d(x,y), \sup_{x\in A_\alpha}\inf_{y\in B_\alpha} d(x,y)\right\}\right\}$$

$$= \sup_{\alpha \in I}\left\{\max\left\{\sup_{y\in B_\alpha}\inf_{x\in A_\alpha} d(y,x), \sup_{x\in A_\alpha}\inf_{y\in B_\alpha} d(x,y)\right\}\right\}$$

$$= d^*(B,A).$$

For the second condition, it is clear that

$$d^*(A,B) \geq 0 \text{ since } d(x,y) \geq 0, \forall\, x, y \in X.$$

Now, we have to show that $d^*(A,B) = 0$ if and only if $A = B$. Suppose that $d^*(A,B) = 0$, but $A \neq B$. Therefore, for $\alpha \in I$, $A_\alpha \neq B_\alpha$. Suppose, there exits $a \in A_\alpha$ and $a \notin B_\alpha$. Consequently, we have:

$$0 < \inf_{y\in B_\alpha} d(a,y) \leq \sup_{x\in A_\alpha}\inf_{y\in B_\alpha} d(x,y)$$

$$\leq \max\left\{ \sup_{x\in A_\alpha} \inf_{y\in B_\alpha} d(x,y), \sup_{y\in B_\alpha} \inf_{x\in A_\alpha} d(y,x) \right\}$$

$$\leq d^*(A,B),$$

which contradicts the fact that $d^*(A,B) = 0$. Hence $A = B$.

For the only if direction, suppose that $A = B$, therefore, $A_\alpha = B_\alpha$, $\forall\ \alpha \in I$, hence $x \in A_\alpha$ implies that $x \in B_\alpha$, and

$$d^*(A,B) = \sup_{\alpha\in I}\left\{ \max\left\{ \sup_{x\in A_\alpha} \inf_{y\in B_\alpha} d(x,y), \sup_{y\in B_\alpha} \inf_{x\in A_\alpha} d(y,x) \right\}\right\}$$

$$= \sup_{\alpha\in I}\left\{ \max\left\{ \sup_{x\in B_\alpha} \inf_{y\in B_\alpha} d(x,y), \sup_{y\in B_\alpha} \inf_{x\in B_\alpha} d(y,x) \right\}\right\}$$

$$= 0.$$

Finally, to prove the third condition

$$d^*(A,B) \leq d^*(A,C) + d^*(C,B),\ \forall\ A, B, C \in X^*$$

Take any $\alpha \in I$, $a \in A_\alpha$, therefore,

$$\inf_{b\in B_\alpha} d(a,b) \leq d(a,b) \leq d(a,c) + d(c,b),\ \forall\ b \in B_\alpha,\ c \in C_\alpha,$$

since there exists $b^* \in B_\alpha$ such that $d(c,b^*) = \inf_{b\in B_\alpha} d(c,b)$. Hence

$$\inf_{b\in B_\alpha} d(a,b) \leq d(a,c) + d(c,b^*)$$

$$= d(a,c) + \inf_{b\in B_\alpha} d(c,b)$$

$$\leq d(a,c) + \sup_{c\in C_\alpha} d(c,b),\ \forall\ c \in C_\alpha$$

and similarly, there exists $c^* \in C_\alpha$ such that $d(a,c^*) = \inf_{b\in C_\alpha} d(a,c)$, therefore,

$$\inf_{b\in B_\alpha} d(a,b) \le \sup_{a\in A_\alpha}\inf_{c\in C_\alpha} d(a,c) + \sup_{c\in C_\alpha}\inf_{b\in B_\alpha} d(c,b)$$

$$\le \max\left\{\sup_{a\in A_\alpha}\inf_{c\in C_\alpha} d(a,c), \sup_{c\in C_\alpha}\inf_{a\in A_\alpha} d(c,a)\right\} + \max\left\{\sup_{c\in C_\alpha}\inf_{b\in B_\alpha} d(c,b), \sup_{b\in B_\alpha}\inf_{c\in C_\alpha} d(b,c)\right\}$$

and since this is true for any $a \in A_\alpha$, $\alpha \in I$, hence there exists $a^* \in A_\alpha$ such that:

$$\inf_{b\in B_\alpha} d(a^*,b) \le \sup_{a\in A_\alpha} d(a,b).$$

Hence

$$\sup_{a\in A_\alpha}\inf_{b\in B_\alpha} d(a,b) \le \max\left\{\sup_{a\in A_\alpha}\inf_{c\in C_\alpha} d(a,c), \sup_{c\in C_\alpha}\inf_{a\in A_\alpha} d(c,a)\right\} + \max\left\{\sup_{c\in C_\alpha}\inf_{b\in B_\alpha} d(c,b), \sup_{b\in B_\alpha}\inf_{c\in C_\alpha} d(b,c)\right\}. \quad (3.2)$$

Similarly, we can show that:

$$\sup_{b\in B_\alpha}\inf_{a\in A_\alpha} d(b,a) \le \max\left\{\sup_{b\in B_\alpha}\inf_{c\in C_\alpha} d(b,c), \sup_{c\in C_\alpha}\inf_{b\in B_\alpha} d(c,b)\right\} + \max\left\{\sup_{c\in C_\alpha}\inf_{a\in A_\alpha} d(c,a), \sup_{a\in A_\alpha}\inf_{c\in C_\alpha} d(a,c)\right\}. \quad (3.3)$$

The right hand side of the inequalities (3.2) and (3.3) are the same, therefore

$$\max\left\{\sup_{a\in A_\alpha}\inf_{b\in B_\alpha} d(c,b), \sup_{b\in B_\alpha}\inf_{a\in A_\alpha} d(b,a)\right\} \le \max\left\{\sup_{a\in A_\alpha}\inf_{c\in C_\alpha} d(a,c), \sup_{c\in C_\alpha}\inf_{a\in A_\alpha} d(c,a)\right\} + \max\left\{\sup_{c\in C_\alpha}\inf_{b\in B_\alpha} d(c,b), \sup_{b\in B_\alpha}\inf_{c\in C_\alpha} d(b,c)\right\}$$

and this is true for any $\alpha \in I$, hence we have:

$$d^*(A,B) \le d^*(A,C) + d^*(C,B)$$

which completes the proof. ❑

As a consequence to the last theorem, the set of all closed and bounded fuzzy subsets X^* of a metric space X form a metric space which is called ***fuzzy metric space***. Inductively, many results and properties about fuzzy metric spaces could be proved. We are interested here about the completeness of the fuzzy metric space (X^*,d^*) whenever the original metric space (X,d) is complete, i.e., the completeness of the space (X^*,d^*) is an inheritable property that could be deduced from the completeness of the metric space (X,d).

Definition 3.3:

Let S be a fuzzy subset of I^X with membership function $\mu_s : X \longrightarrow [0,1]$ and let $\delta > 0$ be any given real number. The δ-neighborhood $S+\delta$ of S in I^X is defined using the α-level set by:

$$(S+\delta)_\alpha = \left\{ y \in X : \exists\ x \in S_\alpha \text{ such that } d^*(x,y) \le \delta,\ \forall \alpha \in I \text{ and } \mu_S(y)+\delta \in [0,1] \right\}$$

where

$$(S+\delta)_\alpha = S_\alpha + \delta,\ \forall\ \alpha \in I$$

It is clear that $S+\delta$ is also a fuzzy subset.

Theorem 3.2:

Suppose that A *is closed and bounded in* X^*. *Then* $A+\varepsilon$ *is also closed and bounded in* X^*, i.e., if A_α is closed and bounded then $A_\alpha+\varepsilon$ is also closed and bounded for each $\alpha \in I$.

Proof:

Choose any real number $M > 0$, then there is some $b \in B_\alpha$, so that

$d^*(a,b) \le M,\ \forall\ a \in A$ and $b \in B \in X^*$,

and let v be an arbitrary element of $A_\alpha + \varepsilon$, then there is an element $a \in A_\alpha$ such that $d^*(v,a) \le \varepsilon$.

Hence,

$$d^*(v,b) \le d^*(v,a) + d^*(a,b) \le \varepsilon + M.$$

Therefore, $A_\alpha + \varepsilon$ is bounded by a real number $M + \varepsilon$. Hence $A + \varepsilon$ is bounded.

Now, to show that $A_\alpha + \varepsilon$ is closed, let us suppose that $\{v_n\}_{n=1}^{\infty}$ any sequence in $A_\alpha + \varepsilon$ that converge to v. We will show that $v \in A_\alpha + \varepsilon$.

For each $n \in \mathbb{N}$, $v_n \in A_n + \varepsilon \longrightarrow \exists\ a_n \in A_n$ such that $d^*(a_n, v_n) \le \varepsilon/3$

Since A is closed and bounded, then using Bolazano-Weierstrass theorem there exists a subsequence $\{a_{n_k}\}_{k=1}$ such that

$$a_{n_k} \longrightarrow a \quad \text{as} \quad k \longrightarrow \infty, \text{ i.e., } d^*(a_{n_k}, a) \le \frac{\varepsilon}{3},$$

also

$$d^*(a,v) \le d^*(a,a_{n_k}) + d^*(a_{n_k}, v_{n_k}) + d^*(v_{n_k}, v)$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

hence $v \in A_\alpha + \varepsilon$, i.e., the limit point for each convergent subsequence in $A_\alpha + \varepsilon$ belongs to $A_\alpha + \varepsilon$, i.e., $A_\alpha + \varepsilon$ is closed. ❑

Similarly as it is pointed in chapter two, the meanings and definitions of convergence, Cauchy sequences, completeness ... etc. becomes meaningful. Therefore, the following lemma is of great importance, since it gives an equivalence definition to the distance meaning between fuzzy sets.

Lemma 3.1:

Let $A, B \in X^*$ *and* $\varepsilon > 0$ *be any given real number. Then* $d^*(A,B) \le \varepsilon$ *if and only if* $A_\alpha \subset B_\alpha + \varepsilon$ *and* $B_\alpha \subset A_\alpha + \varepsilon$, $\forall\ \alpha \in I$.

Proof:

Suppose that $d^*(A,B) \le \varepsilon$, hence for all $\alpha \in I$, we have $\sup_{x\in A_\alpha} \inf_{y\in B_\alpha} d(x,y) \le \varepsilon$ and $\sup_{y\in B_\alpha} \inf_{x\in A_\alpha} d(y,x) \le \varepsilon$,

for the first part, consider $\sup_{x\in A_\alpha} \inf_{y\in B_\alpha} d(x,y) \le \varepsilon$

$\Rightarrow$ for all $x \in A_\alpha$, $\inf_{y\in B_\alpha} d(x,y) \le \varepsilon$

$\Rightarrow$ for all $x \in A_\alpha$ and some $y \in B_\alpha$ we have $d(x,y) \le \varepsilon$

$\Rightarrow$ for all $x \in A_\alpha$ we have $x \in B_\alpha + \varepsilon$

$\Rightarrow A_\alpha \subseteq B_\alpha + \varepsilon$ and smilarly $B_\alpha \subseteq A_\alpha + \varepsilon$.

Conversely, suppose that $A_\alpha \subset B_\alpha + \varepsilon$ and take any $x \in A_\alpha$ which gives $x \in B_\alpha + \varepsilon$

$\Rightarrow \exists\ y \in B_\alpha$ such that $d(x,y) \le \varepsilon$, $\forall\ x \in A_\alpha$

$\Rightarrow \inf_{y\in B_\alpha} d(x,y) \le \varepsilon$, $\forall\ x \in A_\alpha$,

$\Rightarrow \sup_{x\in A_\alpha} \inf_{y\in B_\alpha} d(x,y) \le \varepsilon$,

hence

$$\sup_{x\in A_\alpha} \inf_{y\in B_\alpha} d(x,y) \le \varepsilon \iff A_\alpha \subset B_\alpha + \varepsilon.$$

Similarly, we could prove that

$$\sup_{y\in B_\alpha}\inf_{x\in A_\alpha} d(y,x)\le\varepsilon \Leftrightarrow B_\alpha \subset A_\alpha+\varepsilon.$$

Therefore

$$\sup_{\alpha\in I}\left\{\max\left\{\sup_{x\in A_\alpha}\inf_{y\in B_\alpha} d(x,y), \sup_{y\in B_\alpha}\inf_{x\in A_\alpha} d(y,x)\right\}\right\}\le\varepsilon \Leftrightarrow A_\alpha\subseteq B_\alpha+\varepsilon \text{ and } B_\alpha\subseteq A_\alpha+\varepsilon$$

$$\text{i.e., } d^*(A,B)\le\varepsilon \Leftrightarrow A_\alpha\subseteq B_\alpha+\varepsilon \text{ and } B_\alpha\subseteq A_\alpha+\varepsilon. \qquad (3.4)$$

❑

The above lemma could also be considered as one of the most important properties of α-level sets.

In order to be familiar with Cauchy sequences in fuzzy sets, an alternative definition based on the previous lemma will be given.

Remark 3.1:

A sequence $\{A_n\}_{n=1}^{\infty}$ of fuzzy subsets of X^* is said to be fuzzy Cauchy sequence, if for any real number $\varepsilon > 0$ there is a natural number N such that for all n, $m \ge N$ implies that $d^*(A_n,A_m)\le\varepsilon$ which is equivalent to $A_{n_\alpha}\subset A_{m_\alpha}+\varepsilon$ and $A_{m_\alpha}\subset A_{n_\alpha}+\varepsilon$.

This Cauchy sequence of fuzzy sets have the following property. There is a Cauchy sequence $\{x_n\}_{n=1}^{\infty}\in X$ with the condition that $x_n\in A_n$, $\forall$ n.

We need the following property which allows the extension of a Cauchy subsequences $\{x_{n_j}\}_{j=1}^{\infty}$, with the property that $x_{n_j}\in A_{n_j}$, to a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$, with the property that $x_n\in A_n$, $\forall$ n.